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Mosco-convergence and Wiener measures for conductive thin boundaries ☆

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ABSTRACT

The Mosco-convergence of energy functionals and the convergence of associated Wiener measures are proved for a domain with highly conductive thin boundary. We obtain those results for matrix-valued conductivities and a family of speed measures (measures of the underlying domain). In particular, this family includes the Lebesgue measure and the one which makes the energy functional superposition. The expectation of the displacement of the associated processes close to the boundary goes to $+\infty$ due to the explosion of the conductivity at the limit.

1. Introduction

In this article, we are interested in the problems of the spectral structure and the stochastic process of the energy defined on a domain Ω with highly concentrated conductivity around the boundary $\Sigma = \partial\Omega$. This is an important mathematical model, for instance, in physical and chemical engineering and biology; however, to analyze those problems in practice is quite complicated (see e.g., H. Attouch [1]). The method, which we will use to overcome this difficulty, is the singular perturbation theory: instead of working directly in this setting, we consider an ideal model, that is, the conductivity is concentrated only on the boundary Σ and is constant inside $\Omega \setminus \Sigma$, and apply various results for this ideal model (see e.g., M. Fukushima [7], N. Ikeda and S. Watanabe [10], M. Tomisaki [34], and Y. Ogura et al. [27], etc. We refer the reader to Y. Ogura et al. [26] and the reference within). The crucial step in this method is to justify that those results for the ideal model well describe our original problems. This is the main issue which we will discuss in the current article.

Let us explain our setting. Because we would like to keep our discussions transparent, we let $\Omega = B \times \mathbb{R}$ be the simplest geometry; namely, a cylinder in $\mathbb{R}^3$ with infinite length and the section is the 2-dimensional open unit disc B.[1] The conductivity is represented by the matrix a^ϵ which is diagonal with respect to the bases $\{\partial_r, \partial_\theta, \partial_z\}$ associated to the cylindrical coordinates (r, θ, z) as

$$a^\epsilon = \begin{pmatrix} \epsilon^{-\alpha} & 0 & 0 \\ 0 & \epsilon^{-\beta} & 0 \\ 0 & 0 & \epsilon^{-\gamma} \end{pmatrix}$$

on the ϵ-neighborhood Σ^ϵ of the boundary Σ, and is the identity in $\Omega^{1-\epsilon} = \Omega \setminus \Sigma^\epsilon$. The ideal model corresponds to $\epsilon = 0$ and we will call it the *limit space*. The conductivity goes to positive infinity on Σ^ϵ as $\epsilon \to 0$ if at least one of α, β, γ is

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[1] This is one of the standard models in the study of super conductivity.

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positive, and it appears as a singular term in the energy functional F of the limit space. The associated energy functionals are (see Section 2 for further definition):

$$F^\epsilon[u] = \int_\Omega a^\epsilon[\nabla u]\,d\mu,$$

$$F[u] = \int_\Omega |\nabla u|^2\,d\mu + F_\Sigma[u],$$

where $a^\epsilon[\nabla u] = a^\epsilon(\nabla u, \nabla u)$. [We are postponing all domain questions until the main body of the paper.] Here, $\nabla u = (u_r, u_\theta/r, u_z)$, $d\mu = r\,dr\,d\theta\,dz$ is the 3-dimensional Lebesgue measure, and F_Σ is the energy functional on Σ which depends on β and γ as

$$F_\Sigma[u] = \int_\Sigma a(\nabla u|_\Sigma, \nabla u|_\Sigma)\,d\sigma$$

with

$$a = \begin{pmatrix} \delta_{\{\beta=1\}} & 0 \\ 0 & \delta_{\{\gamma=1\}} \end{pmatrix}$$

with respect to the bases $\{\partial_\theta, \partial_z\}$ and $d\sigma = d\theta\,dz$ is the surface measure on Σ.

In order to study the spectral structure, we prove the Mosco-convergence of F^ϵ to F [21,22]. Indeed, the convergence of the spectral structure is one of the consequences of the Mosco-convergence [23]. The Mosco-convergence for the highly conductive thin layers was initiated in the 1970s by L. Carbone and C. Sbordone [5], H. Pham Huy and E. Sanchez-Palencia [30]. Since then, there have been various extensions (we refer the reader to the book [1] by H. Attouch and the paper [23] by U. Mosco), and the theory was recently extended to quite complicated geometries including fractals and pre-fractals by M.R. Lancia and M.A. Vivaldi [18], U. Mosco and M.A. Vivaldi [24,25], M.R. Lancia et al. [19], and M.R. Lancia and P. Vernole [17]. Our setting and the analytical approach is inspired by those previous works.

For the study of the stochastic process, we show that the Wiener measure $\mathbb{P}^\epsilon$ associated to F^ϵ weak converges to that $\mathbb{P}$ of the limit space. We will prove that $\{\mathbb{P}^\epsilon\}_{\epsilon>0}$ is tight, indeed, the Mosco-convergence implies the convergence of finite-dimensional distributions of $\mathbb{P}^\epsilon$, and together with the tightness, the weak convergence $\mathbb{P}^\epsilon \to \mathbb{P}$ follows. This idea of combining the Mosco-convergence and the tightness to deduce the Wiener measures convergence was pointed out by M. Fukushima in early 1990s, and this method was applied and extended by e.g., T. Uemura [35], K. Kuwae and T. Uemura [15,16], A. Kasue et al. [11], Y. Ogura et al. [28], K. Kuwae and T. Shioya [14], and A. Kolesnikov [12,13].

We investigate those problems with the following speed measures (underlying measures) for Ω:

$$d\mu^\epsilon = \omega^\epsilon\,d\mu,$$

here ω^ϵ is 1 on $\Omega^{1-\epsilon}$ and is $\epsilon^{-\nu}$ on Σ^ϵ with $0 \leqslant \nu \leqslant 1$. We emphasize that F^ϵ is defined naturally in $L^2(\Omega; d\mu^\epsilon)$ and is independent of $d\mu^\epsilon$; however, the associated generators, spectral structures, heat-semigroups, and the stochastic processes depend both on F^ϵ and $d\mu^\epsilon$. The measure $d\mu^\epsilon$ weak converges on $C_0(\overline{\Omega})$ (see Lemma 4.0.4) as

$$d\mu^\epsilon \to d\mu' := d\mu + \delta_{\{\nu,r=1,1\}}\,d\sigma.$$

If $\nu = 0$ then $d\mu^\epsilon \equiv d\mu$ for every $\epsilon > 0$, and F^ϵ converges to the superposition F in $L^2(\Omega; d\mu')$ if and only if $\nu = 1$. For $0 < R \leqslant \infty$, we let $\Omega_R = B \times (-R, R)$, F_R and F_R^ϵ be the restriction of F and F^ϵ on Ω_R with Dirichlet homogeneous boundary condition at $B \times \{-R, R\}$. We drop the R from the notations when $R = \infty$. Our main result is

Main result. Let $R \leqslant \infty$ and F_R^ϵ and F_R be the energy functionals defined in $L^2(\Omega_R; d\mu^\epsilon)$ and $L^2(\Omega_R; d\mu')$ respectively as above. It follows that F_R^ϵ and F_R are local and regular Dirichlet forms.

Assume $R < \infty$. If $\alpha \geqslant 0$, and $0 \leqslant \beta, \gamma, \nu \leqslant 1$, then F_R^ϵ Mosco-converges to F_R. Accordingly, the associated L^2-heat-semigroups $T_R^\epsilon(t)$ and $T_R(t)$ satisfy

$$T_R^\epsilon(t) \to T_R(t) \quad \text{as } \epsilon \to 0 \text{ for every } t > 0,$$

and the associated spectral measures E^ϵ and E satisfy

$$E^\epsilon\big((\lambda, \eta]\big) \to E\big((\lambda, \eta]\big) \quad \text{as } \epsilon \to 0$$

for every $\lambda < \eta$, which are not in the point spectrum. In particular, F_R^ϵ Γ-converges to F_R, and as a consequence, every cluster point of the sequence u^ϵ of the minimizers of F_R^ϵ is a minimizer of F_R. Namely, let $a_m > 0$ and $\epsilon(m) > 0$ be sequences tending to 0 as $m \to \infty$ and $u_m \in L^2(\Omega_R; d\mu^{\epsilon(m)})$ satisfy:

$$F_R^{\epsilon(m)}[u_m] < \inf_{u \in L^2(\Omega_R; d\mu^{\epsilon(m)})} F_R^{\epsilon(m)}[u] + a_m.$$

If $u_{k(m)}$ is a subsequence of u_m converging to some $\bar{u}$ in $L^2(\Omega_R; d\mu^{\epsilon(m)})$, then $\bar{u}$ is a minimizer of F_R and

$$F_R[\bar{u}] = \lim_{m\to 0} F_R^{\epsilon(k(m))}[u_{k(m)}].$$

Assume $R \leqslant \infty$. If

$$\alpha \geqslant 2\max\{\beta, \gamma\} \quad \text{and} \quad 0 \leqslant \nu \leqslant 1, \tag{1}$$

then the set of Wiener measures $\{\mathbb{P}^\epsilon\}_{\epsilon>0}$ associated to F^ϵ is tight.

Furthermore, if $0 \leqslant \beta, \gamma, \nu \leqslant 1$ and $\alpha \geqslant 2\max\{\beta, \gamma\}$, then $\mathbb{P}^\epsilon$ weak converges to the Wiener measure $\mathbb{P}$ associated to F. In particular, F^ϵ converges to F in both Mosco and Γ senses.

By considering the matrix-valued conductivities a^ϵ rather than a scalar-valued conductivity, we find some new phenomena. First, we see that for the Mosco-convergence, $\alpha \geqslant 0$ can be arbitrary large; namely, the energy in the r-direction in F_R^ϵ may be arbitrary large. In order to establish the Mosco-convergence, we need to prove both "upper" and "lower" bound conditions (see Theorem 3.0.4). In particular, for the upper-bound condition, we need to find for every $u \in L^2(\Omega_R; d\mu')$ a sequence of functions $u_\epsilon \in L^2(\Omega_R; d\mu^\epsilon)$ converging to u and $F_R^\epsilon[u_\epsilon]$ is bounded by $F[u]$. Therefore, in general, it is hard to establish the upper-bound condition if F_R^ϵ has high energy. However, we construct the approximating functions u_ϵ (see (8)) which charge no energy in the r-direction, and therefore α can be arbitrary large.

Next, we observe that $\{\mathbb{P}^\epsilon\}_{\epsilon>0}$ is tight with arbitrary α, β, and γ satisfying (1). We may explain this condition as follows. Roughly speaking, in order to prove the tightness, we need to estimate the distance $|X^\epsilon(t)|$ of the process X^ϵ associated to F_R^ϵ at the limit $\epsilon \to 0$. If one of α, β, and γ is large, then X^ϵ moves faster, and, in general, it becomes difficult to establish this estimate. However, in our setting, if X^ϵ exits Σ^ϵ, which is determined by the r-component of X^ϵ (namely, controlled by α), then X^ϵ is a Brownian motion. Thus, even if X^ϵ moves fast, if α is large enough relative to β and γ, then X^ϵ exits Σ^ϵ soon enough, and we can estimate $|X^\epsilon|$. On the contrary, if α is small, then X^ϵ moves slowly in the r-direction, and accordingly, X^ϵ may stay in Σ^ϵ long enough so that it can run in the θ or z-directions to violate the estimate. The lower bound for α as in (1) is needed to avoid this situation. Since Σ^ϵ becomes thinner as $\epsilon \to 0$, X^ϵ exits Σ^ϵ faster for smaller $\epsilon > 0$, and this fact appears as the modified 2-D Bessel local time at 1.

As another issue, we study the following long time behavior of the process. Every (L^2-)semigroup T_t associated to a Dirichlet form can be uniquely extended to a bounded semigroup on L^∞, which we denote by the same symbol, due to the Markov property of T_t. The associated process is called *conservative* or *non-explosive* if

$$T_t 1 \equiv 1 \quad \text{for every } t > 0.$$

This means that the process can be found in Ω for every $t > 0$ almost surely. We will show

Proposition *(Conservation property and recurrence). The processes associated to F^ϵ defined in $L^2(\Omega; d\mu^\epsilon)$, F defined in $L^2(\Omega; d\mu)$ and $L^2(\Omega; d\mu')$ are conservative. More strongly, they are recurrent.*

The conservation property follows from the recurrent property; however we prove those two properties independently. This is because if we consider a more general setting such as $B \times \mathbb{R}^n$ or a Riemannian manifold, then there are many important manifolds whose processes are not recurrent but merely conservative. For instance, if we replace $\mathbb{R}$ in our setting by $\mathbb{R}^n$, then our proof shows that the associated processes are recurrent if and only if $n = 1, 2$; and are conservative for every $n \geqslant 1$.

A direct consequence of the Proposition is the following. According to M. Biroli and U. Mosco [3], we say that $u \in L^1_{loc}$ is F-subharmonic (superharmonic, respectively) if

$$F(u, v) \leqslant 0 \ (\geqslant 0, \text{respectively}) \quad \text{for every } v \in C_0^\infty(\Omega) \text{ with } v \geqslant 0.$$

It follows from the Proposition and results by K.Th. Sturm [33] that

Corollary *(Liouville properties). If u is either*

(1) *non-negative L^1-F-superharmonic, or*
(2) *L^∞-F-subharmonic,*

then u is identically a constant. The same conclusion holds true if we replace F by F^ϵ.

Our approach to the Main result is as follows. We prove the Mosco-convergence

$$F_R^\epsilon \to F_R$$

first for the bounded domain Ω_R (namely, $R < \infty$) with fixed speed measure (Theorem 3.0.4). Indeed, for a bounded domain, we may reduce the Mosco-convergence to the Γ-convergence, which is in general weaker. The proof is a modification of those previous works which we cited above where the conductivity is a scalar (not the matrix as in our setting). We extend this result to the changing speed measures $d\mu^\epsilon$ (Theorem 4.0.3) as in the frame work of Kuwae and Shioya [14]. The proof is to show that the sequences of functions constructed in the case of the fixed measure (Theorem 3.0.4) establish (M1) and (M2) conditions in this new setting.

We prove that the Mosco-convergence with changing speed measures implies the convergence of the finite-dimensional distributions of the associated Wiener measures (Lemma 5.0.9). This is a crucial step toward the convergence of the Wiener measures with changing speed measures, and is a generalization of a well-known fact for the fixed speed measure (see e.g., [6, Theorem 2.5]). We conclude that in general the Mosco-convergence is equivalent to the convergence of the finite-dimensional distributions of the associated Wiener measures for both the fixed and changing speed measures.

Next, we extend those two results of Mosco-convergence for $R < \infty$, which we obtained above, to $R = \infty$ by showing the convergence of the finite-dimensional distributions of $\mathbb{P}^\epsilon$ to that of $\mathbb{P}$ with $R = \infty$ (Lemma 7.0.11). The idea is as the following. By applying an estimation of $|X^\epsilon|$, we let the expectation that $|X^\epsilon|$ exceeds $R > 0$ arbitrarily small uniformly in $\epsilon > 0$ by letting R be large. Therefore, taking into account that the process associated to F_R^ϵ is the same as that associated to F^ϵ before hitting $B \times \{-R, R\}$, we may reduce the problem to the case of $R < \infty$.

Our contribution to the theory of singular perturbation in the current setting is to generalize: first, a scalar conductivity to a matrix one, allowing the coefficient to be arbitrarily large in the sense described above; second, the fixed underlying measures to changing ones; and third, the bounded domain to an unbounded one.

The tightness of $\{\mathbb{P}^\epsilon\}_{\epsilon>0}$ (Theorem 6.0.10) will be obtained by estimating the expectation of the displacement of the processes. The estimate will be obtained in terms of modified Bessel local time at 1. Finally, by combining the tightness together with the convergence of the finite-dimensional distributions of $\mathbb{P}^\epsilon \to \mathbb{P}$, we arrive at the Main result; namely, the weak convergence of $\mathbb{P}^\epsilon \to \mathbb{P}$ (Theorem 7.0.13).

Let us point out that A. Kasue et al. [11] and Y. Ogura et al. [28] studied the similar problems for compact manifolds and Euclidean space where the underlying metrics and measures degenerate. They showed the Mosco-convergence of local Dirichlet forms to non-local forms at the limit space. In the present article, the underlying measures are fixed or explore.

We arrange the article as follows. Section 2 is devoted to the set-up. We also show that the energy functionals F_R and F_R^ϵ with $0 < R \leqslant \infty$ are local regular Dirichlet forms. As a consequence, there exist the associated diffusion processes and Wiener measures, by the Fukushima theorem [8], which we will analyze in the following sections.

In Section 3, we prove the Mosco-convergence $F_R^\epsilon \to F_R$ with fixed measure $d\mu$ (namely, with $\nu = 0$) for the bounded domain, and extend this result to changing speed measures $d\mu^\epsilon$ with $0 \leqslant \nu \leqslant 1$ in Section 4. In the latter setting, we will work in the frame work of K. Kuwae and T. Shioya [14] and recall the necessary notions from [14].

In Section 5, we show that Mosco-convergence with changing speed measures implies the convergence of the finite-dimensional distributions of the associated Wiener measures. In Section 6, we show that $\{\mathbb{P}^\epsilon\}_{\epsilon>0}$ is tight. Finally, in Section 7, we prove the Mosco-convergence $F^\epsilon \to F$ and the weak convergences of the Wiener measures. We will also show the conservation property and the recurrence of the processes.

2. Set-up

In this section, we establish the geometry and energy functionals. We also show that the energy functionals are local regular Dirichlet forms. This implies that there are associated stochastic processes of diffusion type, namely, continuous in time.

Let $\Omega_R = B \times (-R, R) \subset \mathbb{R}^3$ be a cylinder with the cylindrical coordinate system (r, θ, z), where $0 < R \leqslant \infty$ and B is the 2-dimensional open unit disc. For the sake of simplicity, hereafter, we drop R from the notations when $R = \infty$, for instance, we denote Ω for Ω_∞, F for F_∞, etc. Let $0 < \epsilon \leqslant 1$ and denote by B^ϵ the 2-dimensional open disc with radius $\epsilon > 0$ and

$$\Omega_R^\epsilon = B^\epsilon \times (-R, R) \subset \Omega_R,$$
$$\Sigma_R^\epsilon = \Omega_R \setminus \Omega_R^{1-\epsilon},$$
$$\Sigma_R = \partial B \times (-R, R).$$

The speed measures (underlying measures) $d\mu^\epsilon$ and $d\mu'$ on Ω are defined as follows:

$$d\mu^\epsilon = \omega^\epsilon\, d\mu,$$

where $d\mu = r\,dr\,d\theta\,dz$ is the standard 3-dimensional Lebesgue measure and

$$\omega^\epsilon(p) = \begin{cases} 1, & p \in \Omega^{1-\epsilon}; \\ \epsilon^{-\nu}, & p \in \Sigma^\epsilon, \end{cases}$$

with $0 \leqslant \nu \leqslant 1$. The measure $d\mu'$ on the “limit space” is

$$d\mu' = d\mu + \delta_{\{\nu, r=1,1\}}\, d\sigma,$$

where σ is the surface measure on Σ_R.

Let $\alpha, \beta, \gamma \in \mathbb{R}$. The energy functional F_R^ϵ and the "limit energy functional" F_R are defined as follows:

$$F_R^\epsilon[u] = \begin{cases} \int_{\Omega_R^{1-\epsilon}} |\nabla u|^2\, d\mu + \int_{\Sigma_R^\epsilon} (\epsilon^{-\alpha}(u_r)^2 + \epsilon^{-\beta}(\frac{u_\theta}{r})^2 + \epsilon^{-\gamma}(u_z)^2)\, d\mu, & u \in D(F_R^\epsilon), \\ \infty, & u \in L^2(\Omega_R; d\mu^\epsilon) \setminus D(F_R^\epsilon), \end{cases}$$

where

$$D(F_R^\epsilon) = \begin{cases} \{u \in H^1(\Omega_R; d\mu^\epsilon); u|_{z\in\{-R,R\}} = 0\}, & R < \infty; \\ H^1(\Omega; d\mu^\epsilon), & R = \infty. \end{cases}$$

We will often denote by $a^\epsilon[\nabla u]$ the integrand of the second term of F_R^ϵ.

$$F_R[u] = \begin{cases} \int_{\Omega_R} |\nabla u|^2\, d\mu + F_{\Sigma_R}[u], & u \in D(F_R), \\ \infty, & u \in L^2(\Omega_R; d\mu') \setminus D(F_R), \end{cases} \tag{2}$$

where

$$F_{\Sigma_R}[u] = \int_{\Sigma_R} \left(\delta_{\{\beta=1\}} |\partial_\theta u|_\Sigma|^2 + \delta_{\{\gamma=1\}} |\partial_z u|_\Sigma|^2\right) d\sigma,$$

and $D(F_R)$ is the completion of

$$\begin{cases} \tilde{D}_R = \{u \in C^\infty(\overline{\Omega}_R)\colon u|_{z\in\{-R,R\}} = 0\}, & R < \infty; \\ \tilde{D} = C_0^\infty(\overline{\Omega}), & R = \infty, \end{cases} \tag{3}$$

with respect to F_R-1 norm, i.e.,

$$\|u\|_{F_R\text{-}1} = \sqrt{F_R[u]} + \|u\|_{L^2(\Omega_R; d\mu')}.$$

In Proposition 2.0.2, we show that, in particular, if $\beta = 1 = \gamma$, then $D(F_R)$ is

$$\begin{aligned} &\{u \in H^1(\Omega_R; d\mu)\colon u|_{z\in\{-R,R\}} = 0,\ u|_{\Sigma_R} \in H_0^1(\Sigma_R; d\sigma)\}, && R < \infty; \\ &\{u \in H^1(\Omega; d\mu)\colon u|_\Sigma \in H^1(\Sigma; d\sigma)\}, && R = \infty. \end{aligned}$$

We note that F_R^ϵ and F_R are defined independently on the speed measures on Ω_R. Let us recall the discussion and the remark in pp. 372 and 375, respectively from [23]:

Lemma 2.0.1. *If G^ϵ and G are energy functionals defined in the same Hilbert space, then*

(1) *G is closed if and only if it is lower-semicontinuous.*
(2) *If G^ϵ Mosco-converges to G, then G is lower-semicontinuous.*

The second statement extends to the functionals defined in different Hilbert spaces in Kuwae–Shioya sense [14, *Lemma* 2.0.1].

We show

Proposition 2.0.2. *Let $0 < R \leqslant \infty$ and $0 \leqslant \nu \leqslant 1$. The polarizations of $(F_R^\epsilon, D(F_R^\epsilon))$ in $L^2(\Omega_R; d\mu^\epsilon)$ and $(F_R, D(F_R))$ in $L^2(\Omega_R; d\mu)$ and $L^2(\Omega_R; d\mu')$ are local regular Dirichlet forms. In particular, for $\beta = 1 = \gamma$, $D(F_R)$ is*

$$\begin{cases} \{u \in H^1(\Omega_R; d\mu)\colon u|_{z\in\{-R,R\}} = 0,\ u|_{\Sigma_R} \in H_0^1(\Sigma_R; d\sigma)\}, & R < \infty; \\ \{u \in H^1(\Omega; d\mu)\colon u|_\Sigma \in H^1(\Sigma; d\sigma)\}, & R = \infty. \end{cases} \tag{4}$$

Proof. We prove those properties only for $(F, D(F))$ in $L^2(\Omega; d\mu')$ because other cases can be proven in similar ways. (We note that any function u from the completion of $\tilde{D}_R$ with respect to the norm F_R-1 vanishes if $z = \pm R$.) Moreover, since the local property, regularity, and Markov property are clear, we show only that the form is closed and (4).

First, in order to show that F is closed, we prove that F with $C_0^\infty(\overline{\Omega})$ is closable in $L^2(\Omega; d\mu')$; namely, if $u_n \in C_0^\infty(\overline{\Omega})$ satisfy

$$\lim_{n,m\to\infty} F[u_n - u_m] = 0 \quad \text{and} \quad \lim_{n\to\infty} \|u_n\|_{L^2(\Omega; d\mu')} = 0,$$

then $\lim_{n\to\infty} F[u_n] = 0$. By applying [8, Problem 1.1.2], it can be shown that F_Σ is closable with $C_0^\infty(\Sigma)$. By [8, Theorems 3.1.4 and 3.1.5], F with $C_0^\infty(\overline{\Omega})$ is closable in $L^2(\Omega; d\mu)$, so is in $L^2(\Omega; d\mu')$ since $\|u_n\|_{L^2(\Omega; d\mu)} \leqslant \|u_n\|_{L^2(\Omega; d\mu')}$.

Next, we assume $\beta = 1 = \gamma$ and show (4). Denote by $\tilde{D}(F)$ the right-hand side of (4). We show:

$$C_0^\infty(\overline{\Omega}) \text{ is dense in } \tilde{D}(F) \text{ with respect to } F_1\text{-norm.} \tag{5}$$

For $u \in \tilde{D}(F)$ we will find $u^\delta \in C_0^\infty(\overline{\Omega})$ converging to u in F_1-norm. Let $\chi \in C^\infty([0,1])$ be a function of r such that $0 \leqslant \chi(r) \leqslant 1$ for any $0 \leqslant r \leqslant 1$ and

$$\chi(r) = \begin{cases} 0, & 0 \leqslant r \leqslant 1/2; \\ 1, & 3/4 \leqslant r \leqslant 1. \end{cases} \tag{6}$$

Extend $v = u|_\Sigma$ to whole Ω, which we denote by the same symbol v, by

$$v(r,\theta,z) = \begin{cases} 0, & 0 \leqslant r < 1/3; \\ v(\theta,z), & 1/3 \leqslant r \leqslant 1. \end{cases}$$

Set $\bar{v} = \chi v \in H^1(\Omega; d\mu)$. Take $v^\delta \in C_0^\infty(\Sigma)$ converging to v in $H^1(\Sigma; d\sigma)$ and extend it to Ω, which we denote by the same symbol v^δ, as

$$v^\delta(r,\theta,z) = \begin{cases} 0, & 0 \leqslant r < 1/3; \\ v^\delta(\theta,z), & 1/3 \leqslant r \leqslant 1, \end{cases}$$

and set $\bar{v}^\delta$ as

$$\bar{v}^\delta = \chi v^\delta \in C_0^\infty(\overline{\Omega}).$$

Taking into account $(u - \bar{v})|_\Sigma = 0$, implying $u - \bar{v} \in H_0^1(\Omega; d\mu)$, let $u_\delta \in C_0^\infty(\overline{\Omega} \setminus \Sigma)$ converge to $u - \bar{v}$ in $H^1(\Omega; d\mu)$ (note that we may assume that $u_\delta \in C_0^\infty(\overline{\Omega} \setminus \Sigma^\delta)$). Set

$$u^\delta = u_\delta + \bar{v}^\delta \in C_0^\infty(\overline{\Omega}). \tag{7}$$

Since χ has support in $[1/2, 1]$ and $v^\delta \to v$ in $H^1(\Sigma; d\sigma)$, it follows that

$$\begin{aligned} \|\bar{v}^\delta - \bar{v}\|^2_{L^2(\Omega;d\mu)} &= \int_0^1 \int_\Sigma \left|\chi\left(v^\delta - v\right)\right|^2 d\sigma\, r\, dr \\ &\leqslant \int_{1/2}^1 r\, dr \int_\Sigma \left|v^\delta - v\right|^2 d\sigma \\ &\leqslant \left\|v^\delta - v\right\|^2_{L^2(\Sigma;d\sigma)} \to 0 \quad \text{as } \delta \to 0, \end{aligned}$$

and

$$\begin{aligned} \left\|\nabla\left(\bar{v}^\delta - \bar{v}\right)\right\|_{L^2(\Omega;d\mu)} &\leqslant \|\nabla\chi\|_{L^\infty(\Omega)} \left\|v^\delta - v\right\|_{L^2(\Sigma^{1/2};d\mu)} + \left\|\nabla\left(v^\delta - v\right)\right\|_{L^2(\Sigma^{1/2};d\mu)} \\ &\leqslant \|\nabla\chi\|_{L^\infty(\Omega)} \sqrt{\int_{1/2}^1 r\, dr \int_\Sigma \left|v^\delta - v\right|^2 d\sigma} + \sqrt{\int_{1/2}^1 r\, dr \int_\Sigma \left|\nabla\left(v^\delta - v\right)\right|^2 d\sigma} \to 0 \end{aligned}$$

as $\delta \to 0$. Thus, $\bar{v}^\delta \to \bar{v}$ in $H^1(\Omega; d\mu)$. This together with the definition of u_δ, gives

$$\begin{aligned} \left\|u^\delta - u\right\|_{H^1(\Omega;d\mu)} &= \left\|u_\delta + \bar{v}^\delta - u\right\|_{H^1(\Omega;d\mu)} \\ &\leqslant \left\|u_\delta - (u - \bar{v})\right\|_{H^1(\Omega;d\mu)} + \left\|\bar{v}^\delta - \bar{v}\right\|_{H^1(\Omega;d\mu)} \to 0 \quad \text{as } \delta \to 0. \end{aligned}$$

On the other hand,

$$\begin{aligned} \left\|u^\delta\big|_\Sigma - u|_\Sigma\right\|_{H^1(\Sigma;d\sigma)} &= \left\|\bar{v}^\delta\big|_\Sigma - u|_\Sigma\right\|_{H^1(\Sigma;d\sigma)} \\ &= \left\|v^\delta - v\right\|_{H^1(\Sigma;d\sigma)} \to 0 \quad \text{as } \delta \to 0. \end{aligned}$$

Therefore, $u^\delta \to u$ in F_1-norm and (5) is proven. We will later show the Mosco-convergence of F^ϵ to F in Theorem 7.0.14, thus, it follows by Lemma 2.0.1 that $\tilde{D}(F)$ is complete with respect to F_1-norm. □

Remark 2.0.3. We point out that $\tilde{D}_R$ is dense in $L^2(\Omega_R; d\mu')$. This can be seen as follows. Recall that $\nu = 1$ in this case. Let $u \in L^2(\Omega_R; d\mu')$. Pick $\tilde{v}_n \in C_0^\infty(\Sigma_R)$ such that $\tilde{v}_n \to u|_\Sigma$ in $L^2(\Sigma_R; d\sigma)$, and extend it to whole Ω_R, which we denote by the same symbol, as

$$\tilde{v}_n(r,\theta,z) = \begin{cases} \tilde{v}_n(\theta,z), & p \in \Sigma_R^{3/n}; \\ 0, & \text{otherwise}, \end{cases}$$

for $p = (r,\theta,z)$. Let $\chi_n \in C^\infty([0,1])$ be a cut-off function such that $0 \leqslant \chi_n(r) \leqslant 1$ for every $r \in [0,1]$ and $n > 0$, and

$$\chi_n(r) = \begin{cases} 1, & r \in [1-(1/n), 1]; \\ 0, & r \in [0, 1-(2/n)]. \end{cases}$$

Then, $v_n = \chi_n \tilde{v}_n \in \tilde{D}_R$, $v_n \to 0$ in $L^2(\Omega_R; d\mu)$, and $v_n|_\Sigma \to u|_\Sigma$ in $L^2(\Sigma_R; d\sigma)$ as $n \to \infty$. Pick $w_n \in C_0^\infty(\Omega_R)$ such that $w_n \to u$ in $L^2(\Omega_R; d\mu)$. It follows for $u_n = v_n + w_n \in \tilde{D}_R$ that

$$\begin{aligned}&\|u_n - u\|_{L^2(\Omega_R; d\mu')} \\ &\quad \leqslant \|w_n - u\|_{L^2(\Omega_R; d\mu)} + \|v_n\|_{L^2(\Omega_R; d\mu)} + \|v_n|_{\Sigma_R} - u|_{\Sigma_R}\|_{L^2(\Sigma_R; d\sigma)} \to 0\end{aligned}$$

as $n \to \infty$.

3. Mosco-convergence with $\nu = 0$, i.e., the underlying measure is $d\mu$

In this section, we assume $\alpha \geqslant 0$, $0 \leqslant \beta, \gamma \leqslant 1$, and $R < \infty$ to prove the following Mosco-convergence $F_R^\epsilon \to F_R$ with fixed speed measure $d\mu$ on Ω_R:

Theorem 3.0.4. *If $\alpha \geqslant 0$, $0 \leqslant \beta, \gamma \leqslant 1$, $R < \infty$, and $\nu = 0$, then it follows the “upper-bound condition”:*

(M1) *For every $u \in L^2(\Omega_R; d\mu)$, there exists u_ϵ^* converging to u in $L^2(\Omega_R; d\mu)$ with*

$$\limsup_{\epsilon \to 0} F_R^\epsilon\big[u_\epsilon^*\big] \leqslant F_R[u],$$

and the “lower bound condition”:

(M2) *If $v_\epsilon \to v$ weakly in $L^2(\Omega_R; d\mu)$, then*

$$\liminf_{\epsilon \to 0} F_R^\epsilon[v_\epsilon] \geqslant F_R[v].$$

Namely, F_R^ϵ Mosco-converges to F_R as $\epsilon \to 0$.

We prove the theorem in the following subsections.

3.1. (M1) condition

Proposition 3.1.1. (M1) *condition holds true under the condition in Theorem* 3.0.4.

Proof of (M1). We first prove the assertion for $u \in \tilde{D}_R = \{u \in C^\infty(\overline{\Omega}_R)\colon\ u|_{z \in \{-R,R\}} = 0\}$. Define $u_\epsilon(p)$ for $p = (r,\theta,z)$ by

$$u_\epsilon(p) = \begin{cases} u(p), & p \in \Omega_R^{1-2\epsilon}; \\ (r/\epsilon)[u(1,\theta,z) - u(1-2\epsilon,\theta,z)] + (1/\epsilon - 1)u(1-2\epsilon,\theta,z) & \\ \quad + (2 - 1/\epsilon)u(1,\theta,z), & p \in \Omega_R^{1-\epsilon} \setminus \Omega_R^{1-2\epsilon}; \\ u(1,\theta,z), & p \in \Sigma_R^\epsilon. \end{cases} \tag{8}$$

The function u_ϵ is defined on $\Omega_R^{1-\epsilon} \setminus \Omega_R^{1-2\epsilon}$ as a linear interpolation, and it belongs to $H^1(\Omega_R; d\mu)$. Since $\|u_\epsilon\|_{L^\infty} \leqslant \|u\|_{L^\infty}$ for every $\epsilon > 0$ and $\mu(\Sigma_R^{2\epsilon}) \to 0$ as $\epsilon \to 0$,

$$\|u_\epsilon - u\|^2_{L^2(\Omega_R; d\mu)} = \int\limits_{\Sigma_R^{2\epsilon}} (u_\epsilon - u)^2 \, d\mu = \|u\|^2_{L^\infty} \mu\big(\Sigma_R^{2\epsilon}\big) \to 0 \quad \text{as } \epsilon \to 0. \tag{9}$$

Next we show:

$$\lim_{\epsilon\to 0} F_R^\epsilon[u_\epsilon] = F_R[u],$$

$$F_R^\epsilon[u_\epsilon] = \int_{\Omega_R^{1-2\epsilon}} |\nabla u|^2\,d\mu + \int_{\Omega_R^{1-\epsilon}\setminus\Omega_R^{1-2\epsilon}} |\nabla u_\epsilon|^2\,d\mu + \int_{\Sigma_R^\epsilon} a^\epsilon[\nabla u_\epsilon]\,d\mu. \tag{10}$$

The first term in (10) converges as

$$\int_{\Omega_R^{1-2\epsilon}} |\nabla u|^2\,d\mu \to \int_{\Omega_R} |\nabla u|^2\,d\mu, \quad \text{as } \epsilon \to 0.$$

We estimate the second term in (10). For that purpose we prove

Claim. *If $p \in \Omega_R^{1-\epsilon} \setminus \Omega_R^{1-2\epsilon}$, then $|(u_\epsilon)_r|$, $|(u_\epsilon)_\theta|$, and $|(u_\epsilon)_z|$ are uniformly bounded in $\epsilon > 0$.*

In fact, noting $|u_r(r,\theta,z)|$ is bounded,

$$\big|(u_\epsilon)_r\big| = \frac{|u(1,\theta,z) - u(1-2\epsilon,\theta,z)|}{\epsilon} \leqslant (1/\epsilon)\int_{1-2\epsilon}^{1} \big|u_r(r,\theta,z)\big|\,dr \leqslant 2\|u_r\|_{L^\infty}.$$

Next, since u_ϵ can be expressed as

$$u_\epsilon(p) = \left(\frac{r-(1-2\epsilon)}{\epsilon}\right)u(1,\theta,z) + \left(\frac{1-\epsilon-r}{\epsilon}\right)u(1-2\epsilon,\theta,z)$$

and

$$0 \leqslant \frac{r-(1-2\epsilon)}{\epsilon} \leqslant 1 \quad \text{and} \quad 0 \leqslant \frac{1-\epsilon-r}{\epsilon} \leqslant 1,$$

it follows that

$$\begin{aligned}
\big|(u_\epsilon)_\theta(p)\big| &\leqslant \left(\frac{r-(1-2\epsilon)}{\epsilon}\right)\big|u_\theta(1,\theta,z)\big| + \left(\frac{1-\epsilon-r}{\epsilon}\right)\big|u_\theta(1-2\epsilon,\theta,z)\big| \\
&\leqslant \|u_\theta\|_{L^\infty}\left(\frac{r-(1-2\epsilon)}{\epsilon} + \frac{1-\epsilon-r}{\epsilon}\right) \\
&= \|u_\theta\|_{L^\infty}.
\end{aligned}$$

In a similar way, $|(u_\epsilon)_z|$ can be estimated and we conclude the Claim.

The second term in (10) can be estimated by applying the Claim as follows:

$$\begin{aligned}
\int_{\Omega_R^{1-\epsilon}\setminus\Omega_R^{1-2\epsilon}} |\nabla u_\epsilon|^2\,d\mu &\leqslant C(u)\int_{\Sigma_R} d\sigma \int_{1-2\epsilon}^{1-\epsilon} \big(1+r^{-2}\big)r\,dr \\
&\leqslant C(u,R)\big[\epsilon^2 + [\ln r]_{1-2\epsilon}^{1-\epsilon}\big] \to 0, \quad \text{as } \epsilon \to 0.
\end{aligned}$$

The third term in (10) is estimated as:

$$\begin{aligned}
\int_{\Sigma_R^\epsilon} a^\epsilon[\nabla u_\epsilon]\,d\mu &= \int_{\Sigma_R}\int_{1-\epsilon}^{1} \big[\epsilon^{-\beta}\big|u_\theta(1,\theta,z)/r\big|^2 + \epsilon^{-\gamma}\big|u_z(1,\theta,z)\big|^2\big]r\,dr\,d\sigma \\
&= -\frac{\ln(1-\epsilon)}{\epsilon^\beta}\int_{\Sigma_R} |\partial_\theta u|_\Sigma|^2\,d\sigma + \left(\frac{\epsilon-\epsilon^2/2}{\epsilon^\gamma}\right)\int_{\Sigma_R} |\partial_z u|_\Sigma|^2\,d\sigma \\
&\to \int_{\Sigma_R} \big[\delta_{\{\beta=1\}}|\partial_\theta u|_\Sigma|^2 + \delta_{\{\gamma=1\}}|\partial_z u|_\Sigma|^2\big]\,d\sigma \\
&= F_{\Sigma_R}[u] \quad \text{as } \epsilon \to 0
\end{aligned}$$

by L'Hospital's rule. Combining these three estimations, we deduce

$$\lim_{\epsilon\to 0} F_R^\epsilon[u_\epsilon] = \int_{\Omega_R} |\nabla u|^2\, d\mu + F_{\Sigma_R}[u] = F_R[u]. \tag{11}$$

Next we consider $u \in D(F_R)$. By the definition, there exists $u^\delta \in \tilde{D}_R$ such that

$$\begin{cases} u^\delta \to u \quad \text{in } L^2(\Omega_R; d\mu), \\ F_R\big[u^\delta\big] \to F_R[u], \end{cases} \tag{12}$$

as $\delta \to 0$. For each u^δ we find by (9) and (11) a function $u_\epsilon^\delta \in H^1(\Omega_R; d\mu)$ satisfying that

$$\begin{cases} u_\epsilon^\delta \to u^\delta \quad \text{in } L^2(\Omega_R; d\mu), \\ F_R^\epsilon\big[u_\epsilon^\delta\big] \to F_R\big[u^\delta\big], \end{cases} \tag{13}$$

as $\epsilon \to 0$. Now we apply the diagonalization formula [1, Corollary 1.16]: there exists a strictly increasing mapping $\epsilon \to \delta(\epsilon)$ such that $\lim_{\epsilon\to 0} \delta(\epsilon) = 0$ and

$$\limsup_{\epsilon\to 0} F_R^\epsilon\big[u_\epsilon^{\delta(\epsilon)}\big] \leqslant \limsup_{\delta\to 0}\Big(\limsup_{\epsilon\to 0} F_R^\epsilon\big[u_\epsilon^\delta\big]\Big) = F_R[u].$$

By setting $u_\epsilon^* = u_\epsilon^{\delta(\epsilon)}$, we proved (M1) for $u \in D(F_R)$.

Finally, if $u \in L^2(\Omega; d\mu) \setminus D(F_R)$, then by letting $u_\epsilon \equiv u$, we conclude the proof of (M1). $\square$

Remark 3.1.2. The same conclusion holds true for $R = \infty$, if we replace $\tilde{D}_R$ by $C_0^\infty(\overline{\Omega})$ in the proof.

3.2. (M2) condition

Let $v_\epsilon \to v$ weakly in $L^2(\Omega_R; d\mu)$ with $\liminf_{\epsilon\to 0} F_R^\epsilon[v_\epsilon] < \infty$. We may assume without loss of generality that

$$v_\epsilon \to v \quad \text{weakly in } H^1(\Omega_R; d\mu) \text{ and strongly in } L^2(\Omega_R; d\mu).$$

Indeed, since α, β, and γ are non-negative, F_R^ϵ-1 norm is not less than the Sobolev norm in $H^1(\Omega_R; d\mu)$, we may assume that v_ϵ is uniformly bounded in $H^1(\Omega_R; d\mu)$ without loss of generality (recall that v_ϵ does not need to be in $H_0^1(\Omega_R; d\mu)$ because we define it to vanish if $z = \pm R$ but we are not imposing a boundary condition for the entire $\partial\Omega_R$). Hence, there is a subsequence of v_ϵ converging weakly in $H^1(\Omega_R; d\mu)$ and strongly to some $v^* \in L^2(\Omega_R; d\mu)$ by the Rellich–Kondrachov theorem. Since $v^* = v$, it follows the assertion.

For $\delta > 0$ there exists $\epsilon' > 0$ such that for $0 < \epsilon < \epsilon'$

$$\int_{\Sigma_R^\epsilon} |\nabla v|^2\, d\mu < \delta.$$

Because F is weakly lower-semicontinuous in $H^1(\Omega_R; d\mu)$ (in particular, in $H^1(\Omega_R \setminus \Sigma_R^{\epsilon'}; d\mu)$),

$$\begin{aligned}\liminf_{\epsilon\to 0} \int_{\Omega_R\setminus\Sigma_R^\epsilon} |\nabla v_\epsilon|^2\, d\mu &\geqslant \liminf_{\epsilon\to 0} \int_{\Omega_R\setminus\Sigma_R^{\epsilon'}} |\nabla v_\epsilon|^2\, d\mu \\ &\geqslant \int_{\Omega_R\setminus\Sigma_R^{\epsilon'}} |\nabla v|^2\, d\mu \\ &\geqslant \int_{\Omega_R} |\nabla v|^2\, d\mu - \delta.\end{aligned}$$

Because $\delta > 0$ in the above inequality is arbitrary, it follows that

$$\begin{aligned}\liminf_{\epsilon\to 0} F_R^\epsilon[v_\epsilon] &\geqslant \liminf_{\epsilon\to 0} \int_{\Omega_R\setminus\Sigma_R^\epsilon} |\nabla v_\epsilon|^2\, d\mu + \liminf_{\epsilon\to 0} \int_{\Sigma_R^\epsilon} a^\epsilon[\nabla v_\epsilon]\, d\mu \\ &\geqslant \int_{\Omega_R} |\nabla v|^2\, d\mu + \liminf_{\epsilon\to 0} \int_{\Sigma_R^\epsilon} a^\epsilon[\nabla v_\epsilon]\, d\mu.\end{aligned}$$

Therefore, it suffices for (M2) to prove

Lemma 3.2.1. *If v_ϵ converges to v weakly in $H^1(\Omega_R; d\mu)$ and strongly in $L^2(\Omega_R; d\mu)$, then*

$$\liminf_{\epsilon\to 0} \int_{\Sigma_R^\epsilon} a^\epsilon[\nabla v_\epsilon]\,d\mu \geqslant F_{\Sigma_R}[v].$$

Proof. We assume that $v_\epsilon \in \tilde{D}_R = \{u \in C_0^\infty(\overline{\Omega}_R)\colon\ u|_{z\in\{-R,R\}} = 0\}$ without loss of generality. Indeed, there exists $u_\epsilon \in \tilde{D}_R$ such that $\|u_\epsilon - v_\epsilon\|_{F_R\text{-}1} \to 0$ as $\epsilon \to 0$, and it follows that u_ϵ converges to v weakly in $H^1(\Omega_R; d\mu)$ and strongly in $L^2(\Omega_R; d\mu)$ and that

$$\liminf_{\epsilon\to 0} \int_{\Sigma_R^\epsilon} a^\epsilon[\nabla u_\epsilon]\,d\mu = \liminf_{\epsilon\to 0} \int_{\Sigma_R^\epsilon} a^\epsilon[\nabla v_\epsilon]\,d\mu.$$

Set

$$\tilde{v}_\epsilon(\theta, z) = \frac{1}{\epsilon}\int_{1-\epsilon}^{1} v_\epsilon(r,\theta,z)r\,dr \in H_0^1(\Sigma_R).$$

Since

$$\begin{aligned}
\left|(\tilde{v}_\epsilon)_\theta\right|^2 &= \frac{1}{\epsilon^2}\left(\int_{1-\epsilon}^{1} (v_\epsilon)_\theta r\,dr\right)^2 \leqslant \frac{1}{\epsilon^2}\int_{1-\epsilon}^{1} r\,dr \int_{1-\epsilon}^{1} \left|(v_\epsilon)_\theta\right|^2 r\,dr \\
&\leqslant \frac{2\epsilon - \epsilon^2}{2\epsilon^2}\int_{1-\epsilon}^{1} \left|(v_\epsilon)_\theta\right|^2 r\,dr \\
&\leqslant \frac{1}{\epsilon}\int_{1-\epsilon}^{1} \left|(v_\epsilon)_\theta\right|^2 r\,dr
\end{aligned}$$

and, by the similar way,

$$\left|(\tilde{v}_\epsilon)_z\right|^2 \leqslant \frac{1}{\epsilon}\int_{1-\epsilon}^{1} \left|(v_\epsilon)_z\right|^2 r\,dr,$$

it follows that

$$\begin{aligned}
\int_{\Sigma_R^\epsilon} a^\epsilon[\nabla v_\epsilon]\,d\mu &\geqslant \frac{1}{\epsilon}\int_{\Sigma_R^\epsilon} \left(\epsilon^{1-\beta}\left|(v_\epsilon)_\theta\right|^2 + \epsilon^{1-\gamma}\left|(v_\epsilon)_z\right|^2\right)d\mu \\
&\geqslant \frac{1}{\epsilon}\int_{\Sigma_R} d\sigma \int_{1-\epsilon}^{1} \left(\delta_{\{\beta=1\}}\left|(v_\epsilon)_\theta\right|^2 + \delta_{\{\gamma=1\}}\left|(v_\epsilon)_z\right|^2\right) r\,dr \\
&\geqslant \int_{\Sigma_R} \left(\delta_{\{\beta=1\}}\left|(\tilde{v}_\epsilon)_\theta\right|^2 + \delta_{\{\gamma=1\}}\left|(\tilde{v}_\epsilon)_z\right|^2\right)d\sigma \\
&= F_{\Sigma_R}[\tilde{v}_\epsilon].
\end{aligned}$$

We need to show:

$$\liminf_{\epsilon\to 0} F_{\Sigma_R}[\tilde{v}_\epsilon] \geqslant F_{\Sigma_R}[v].$$

Because $\tilde{v}_\epsilon$ is uniformly bounded and F_{Σ_R} is weakly lower-semicontinuous in $H^1(\Sigma_R; d\sigma)$, it suffices to prove that

$$\tilde{v}_\epsilon \to v|_{\Sigma_R} \quad \text{weakly in } L^2(\Sigma_R; d\sigma), \ \text{as } \epsilon \to 0.$$

To the end, we show this.

Since $v_\epsilon \to v$ weakly in $H^1(\Omega_R; d\mu)$ it follows that $v_\epsilon|_{\Sigma_R} \to v|_{\Sigma_R}$ weakly in $L^2(\Sigma_R; d\sigma)$ as $\epsilon \to 0$. By this together with

$$\|\tilde{v}_\epsilon - v_\epsilon\|_{L^2(\Sigma_R; d\sigma)} \to 0, \quad \text{as } \epsilon \to 0 \tag{14}$$

we deduce that $\tilde{v}_\epsilon \to v|_\Sigma$ weakly in $L^2(\Sigma_R; d\sigma)$. We show (14). Set

$$\hat{v}_\epsilon(\theta, z) = \frac{1}{\epsilon}\int_{1-\epsilon}^{1} v_\epsilon(r, \theta, z)\, dr.$$

By applying the Cauchy–Schwarz inequality twice,

$$\begin{aligned}
\|\hat{v}_\epsilon - v_\epsilon|_{\Sigma_R}\|^2_{L^2(\Sigma_R; d\sigma)} &= \int_{\Sigma_R}\left[\frac{1}{\epsilon}\int_{1-\epsilon}^{1} v_\epsilon(r, \theta, z) - v_\epsilon(1, \theta, z)\, dr\right]^2 d\sigma \\
&\leqslant \frac{1}{\epsilon}\int_{\Sigma_R}\int_{1-\epsilon}^{1} \left|v_\epsilon(r, \theta, z) - v_\epsilon(1, \theta, z)\right|^2 dr\, d\sigma \\
&\leqslant \frac{1}{\epsilon}\int_{\Sigma_R}\int_{1-\epsilon}^{1} \left(\int_r^1 \left|(v_\epsilon)_r(s, \theta, z)\right| ds\right)^2 dr\, d\sigma \\
&\leqslant \frac{1}{\epsilon}\int_{\Sigma_R}\int_{1-\epsilon}^{1} \left[(1-r)\int_r^1 |\nabla v_\epsilon|^2\, ds\right] dr\, d\sigma \\
&\leqslant \frac{1}{\epsilon}\int_{1-\epsilon}^{1} (1-r)\left[\int_r^1\int_{\Sigma_R} \left|\nabla v_\epsilon(s, \theta, z)\right|^2 d\sigma\, ds\right] dr. \qquad (15)
\end{aligned}$$

Since r and s in (15) satisfy $1-\epsilon \leqslant r \leqslant 1$ and $r \leqslant s \leqslant 1$, it follows for $0 < \epsilon < 1/2$ that $1/2 < s \leqslant 1$, and hence

$$\int_r^1\int_{\Sigma_R} \left|\nabla v_\epsilon(s, \theta, z)\right|^2 d\sigma\, ds \leqslant 2\int_{\Sigma_R^r} \left|\nabla v_\epsilon(s, \theta, z)\right|^2 d\mu \leqslant 2\|\nabla v_\epsilon\|^2_{L^2(\Omega_R; d\mu)}.$$

Thus, (15) is not greater than

$$\frac{2}{\epsilon}\int_{1-\epsilon}^{1} (1-r)\|\nabla v_\epsilon\|^2_{L^2(\Omega_R; d\mu)}\, dr = \epsilon\|\nabla v_\epsilon\|^2_{L^2(\Omega_R; d\mu)},$$

which tends to 0 as $\epsilon \to 0$ because $v_\epsilon \to v$ weakly in $H^1(\Omega_R; d\mu)$, in particular, $\limsup_{\epsilon\to 0}\|\nabla v_\epsilon\|_{L^2(\Omega_R; d\mu)} < \infty$.

On the other hand, since $v_\epsilon|_{\Sigma_R} \to v|_{\Sigma_R}$ weakly in $L^2(\Sigma_R; d\sigma)$, there exists $C(R) > 0$ such that

$$\begin{aligned}
\|\tilde{v}_\epsilon - \hat{v}_\epsilon|_{\Sigma_R}\|^2_{L^2(\Sigma_R; d\sigma)} &= \int_{\Sigma_R}\left(\frac{1}{\epsilon}\int_{1-\epsilon}^{1} v_\epsilon(1-r)\, dr\right)^2 d\sigma \\
&\leqslant \frac{1}{\epsilon^2}\int_{\Sigma_R}\int_{1-\epsilon}^{1} v_\epsilon^2\, dr\int_{1-\epsilon}^{1} (1-r)^2\, dr\, d\sigma \\
&\leqslant \frac{C(R)}{\epsilon^2}\int_{\Sigma_R}\int_{1-\epsilon}^{1} (1-r)^2\, dr\, d\sigma \\
&= C(R)\int_{\Sigma_R}\left(\frac{1}{\epsilon^2}\int_0^{\epsilon} t^2\, dt\right) d\sigma \to 0, \quad \text{as } \epsilon \to 0.
\end{aligned}$$

Finally, by applying the triangle inequality, we obtain (14) and arrive at the conclusion. □

4. Mosco-convergence with $0 \leqslant \nu \leqslant 1$, i.e., the speed measures $d\mu^\epsilon$ are changing

In this section, we extend the Mosco-convergence with fixed underlying measure, which we proved in the previous section, to changing speed measures $d\mu^\epsilon$ on bounded domain Ω_R. We need to prove both (M1) and (M2) conditions in this new setting. The idea to prove (M1) is to show that the sequences of functions which we constructed to prove (M1) in the previous settings work well with this new setting. The approach to (M2) is to show that the weak convergence in this new setting implies the weak convergence in the classical sense, and to apply the (M2) result from the previous section.

Let us begin by recalling the necessary notions from K. Kuwae and T. Shioya [14] in which the frame work of Mosco-convergence of energy functionals defined in different Hilbert spaces was introduced, restated in the current setting:

Definition 1. (See [14].) Let $0 < R \leqslant \infty$. The space $L^2(\Omega_R; d\mu^\epsilon)$ *converges to* $L^2(\Omega_R; d\mu')$ as $\epsilon \to 0$ if

$$\lim_{\epsilon\to 0} \|u\|_{L^2(\Omega_R;d\mu^\epsilon)} = \|u\|_{L^2(\Omega_R;d\mu')}$$

for every $u \in C = C_0(\overline{\Omega}_R)$ ($C = C(\overline{\Omega}_R)$ if $R < \infty$ and $C = C_0(\overline{\Omega})$ if $R = \infty$). A sequence $u_\epsilon \in L^2(\Omega_R; d\mu^\epsilon)$ *converges strongly* to $u \in L^2(\Omega_R; d\mu')$ as $\epsilon \to 0$ if there exists $\tilde{u}_\delta \in C$ such that

$$\lim_{\delta\to 0} \|\tilde{u}_\delta - u\|_{L^2(\Omega_R;d\mu')} = 0 \quad \text{and} \quad \lim_{\delta\to 0} \limsup_{\epsilon\to 0} \|\tilde{u}_\delta - u_\epsilon\|_{L^2(\Omega_R;d\mu^\epsilon)} = 0.$$

A sequence $u_\epsilon \in L^2(\Omega_R; d\mu^\epsilon)$ *converges weakly* to $u \in L^2(\Omega_R; d\mu')$ as $\epsilon \to 0$ if

$$\lim_{\epsilon\to 0} (u_\epsilon, v_\epsilon)_{L^2(\Omega_R;d\mu^\epsilon)} = (u, v)_{L^2(\Omega_R;d\mu)}$$

for every strong convergence sequence $v_\epsilon \in L^2(\Omega_R; d\mu^\epsilon)$ to $v \in L^2(\Omega_R; d\mu)$. A sequence of energy functionals F_R^ϵ on $L^2(\Omega_R; d\mu^\epsilon)$ *Mosco-converges* to an energy functional F_R on $L^2(\Omega_R; d\mu')$ as $\epsilon \to 0$ if

(M1) for every $u \in L^2(\Omega_R; d\mu^\epsilon)$ there exists $u_\epsilon \in L^2(\Omega_R; d\mu^\epsilon)$ converging strongly to u such that

$$\limsup_{\epsilon\to 0} F_R^\epsilon[u_\epsilon] \leqslant F_R[u];$$

(M2) for every $v_\epsilon \in L^2(\Omega_R; d\mu^\epsilon)$ converging weakly to $u \in L^2(\Omega_R; d\mu')$,

$$\liminf_{\epsilon\to 0} F_R^\epsilon[v_\epsilon] \geqslant F_R[u].$$

Let T^ϵ with $\epsilon > 0$ and T be bounded operators on $L^2(\Omega_R; d\mu^\epsilon)$ and $L^2(\Omega_R; d\mu')$, respectively. T^ϵ *converges to* T as $\epsilon \to 0$ if

$$T^\epsilon u_\epsilon \in L^2\big(\Omega_R; d\mu^\epsilon\big) \text{ converges strongly to } Tu \in L^2\big(\Omega_R; d\mu'\big) \quad \text{as } \epsilon \to 0$$

for every $u_\epsilon \in L^2(\Omega_R; d\mu^\epsilon)$ converging strongly to $u \in L^2(\Omega_R; d\mu')$.

The Mosco theorem [23, Theorem 2.4.1] holds true in this setting:

Lemma 4.0.2. *(See [14]. See also [12].) Let $0 < R \leqslant \infty$, F_R^ϵ and F_R be energy functionals in $L^2(\Omega_R; d\mu^\epsilon)$ and $L^2(\Omega_R; d\mu')$, respectively, and $T_R^\epsilon(t)$ and $T_R(t)$ be the associated semigroups, respectively. F_R^ϵ Mosco-converges to F if and only if $T_R^\epsilon(t)$ converges to $T_R(t)$.*

The main result in this section is

Theorem 4.0.3. *Let $\alpha \geqslant 0$, $0 \leqslant \beta, \gamma \leqslant 1$, and $R < \infty$. F_R^ϵ defined in $L^2(\Omega_R; d\mu^\epsilon)$ Mosco-converges to F_R defined in $L^2(\Omega_R; d\mu')$ as $\epsilon \to 0$.*

We prove this theorem in the following subsections.

4.0.1. Proof for (M1) condition

Before starting the proof, let us show

Lemma 4.0.4. *Let $0 < R < \infty$. It follows*:

(1) *$L^2(\Omega_R; d\mu^\epsilon)$ converges to $L^2(\Omega_R; d\mu')$, as $\epsilon \to 0$.*

(2) *If* $u_\epsilon \in L^2(\Omega_R; d\mu^\epsilon) \to u \in L^2(\Omega_R; d\mu')$ *strongly, then*

$$\int_{\Omega_R} u_\epsilon \, d\mu^\epsilon \to \int_{\Omega_R} u \, d\mu', \quad \textit{as } \epsilon \to 0.$$

Proof. For (1), we show:

$$\int_{\Omega_R} u \, d\mu^\epsilon \to \int_{\Omega_R} u \, d\mu', \quad \text{as } \epsilon \to 0 \text{ for every } u \in C. \tag{16}$$

Recall that $d\mu' = d\mu + \delta_{\{\nu=1,\ r=1\}} \, d\sigma$. First, let $\nu < 1$.

$$\begin{aligned}
\left| \int_{\Sigma_R^\epsilon} u \, d\mu^\epsilon \right| &\leqslant \int_{\Sigma_R} \int_{1-\epsilon}^{1} |u| \epsilon^{-\nu} r \, dr \, d\sigma \\
&\leqslant C(R) \|u\|_{L^\infty(\overline{\Omega}_R)} \int_{1-\epsilon}^{1} \epsilon^{-\nu} r \, dr \\
&= C(R) \|u\|_{L^\infty(\overline{\Omega}_R)} \left(\frac{\epsilon^{1-\nu}(\epsilon+2)}{2} \right) \to 0, \quad \text{as } \epsilon \to 0.
\end{aligned}$$

This shows that

$$\int_{\Omega_R} u \, d\mu^\epsilon = \int_{\Omega_R^{1-\epsilon}} u \, d\mu + \int_{\Sigma_R^\epsilon} u \, d\mu^\epsilon \to \int_{\Omega_R} u \, d\mu = \int_{\Omega_R} u \, d\mu', \quad \text{as } \epsilon \to 0.$$

Next, let $\nu = 1$. For an arbitrary $\delta > 0$, let $\epsilon \in (0,1)$ be such that

$$\sup_{(\theta,z) \in \Sigma_R} \left| u(r,\theta,z) - \frac{2}{(2-\epsilon)} u(1,\theta,z) \right| < \delta$$

for $r \in [1-\epsilon, 1]$. Since

$$\int_{\Sigma_R^\epsilon} u(1,\theta,z) \, d\mu = \int_{1-\epsilon}^{1} r \, dr \int_{\Sigma_R} u|_{\Sigma_R} \, d\sigma = \frac{\epsilon(2-\epsilon)}{2} \int_{\Sigma_R} u|_{\Sigma_R} \, d\sigma,$$

it follows that

$$\begin{aligned}
\left| \int_{\Sigma_R^\epsilon} u \, d\mu^\epsilon - \int_{\Sigma_R} u|_{\Sigma_R} \, d\sigma \right| &\leqslant \int_{\Sigma_R^\epsilon} \epsilon^{-1} \left| u(r,\theta,z) - \frac{2}{(2-\epsilon)} u(1,\theta,z) \right| d\mu \\
&\leqslant C(R) \delta \int_{1-\epsilon}^{1} \epsilon^{-1} r \, dr \\
&= C(R) \delta \left(\frac{2-\epsilon}{2} \right).
\end{aligned}$$

Thus,

$$\int_{\Omega_R} u \, d\mu^\epsilon = \int_{\Omega_R^{1-\epsilon}} u \, d\mu + \int_{\Sigma_R^\epsilon} u \, d\mu^\epsilon \to \int_{\Omega_R} u \, d\mu + \int_{\Sigma_R} u|_{\Sigma_R} \, d\sigma = \int_{\Omega_R} u \, d\mu',$$

as $\epsilon \to 0$. The assertion (1) is proved.

The assertion (2) holds true since $1 \in C$ converges strongly to $1 \in L^2(\Omega_R; d\mu')$, and the strong convergence in Kuwae–Shioya sense implies the weak convergence in Kuwae–Shioya sense by [14, Lemma 2.4.1]. □

Proposition 4.0.5 *((M1) condition). Under the same condition in Theorem 4.0.3, for every $u \in L^2(\Omega_R; d\mu')$ there exists $u^\epsilon \in L^2(\Omega_R; d\mu^\epsilon)$ which converges strongly to u and*

$$\limsup_{\epsilon\to 0} F_R^\epsilon\big[u^\epsilon\big] \leqslant F_R[u].$$

Proof. For $u \in D(F_R)$, we will find $\tilde{u}^\delta \in C$ and $u_\epsilon \in D(F_R)$ which satisfy three conditions:

(i) $\lim_{\delta\to 0}\|\tilde{u}^\delta - u\|_{L^2(\Omega_R;d\mu')} = 0,$

(ii) $\lim_{\delta\to 0}\limsup_{\epsilon\to 0}\|\tilde{u}^\delta - u_\epsilon\|_{L^2(\Omega_R;d\mu^\epsilon)} = 0,$

(iii) $\lim_{\epsilon\to 0} F_R^\epsilon[u_\epsilon] \leqslant F_R[u].$

By the definition of $\tilde{D}_R$, there exists $u^\delta \in \tilde{D}_R$ such that

$$\begin{cases} u^\delta \to u \quad \text{in } L^2\big(\Omega_R; d\mu'\big); \\ F_R\big[u^\delta\big] \to F_R[u], \end{cases} \tag{17}$$

as $\delta \to 0$. As (8) in the proof of (M1) in Theorem 3.0.4, we construct u_ϵ^δ from u^δ as an "interpolation"; namely, if $p = (r, \theta, z)$, then

$$u_\epsilon^\delta(p) = \begin{cases} u^\delta(p), & p \in \Omega_R^{1-2\epsilon}; \\ (\frac{r-(1-2\epsilon)}{\epsilon})u^\delta(1,\theta,z) + (\frac{1-\epsilon-r}{\epsilon})u^\delta(1-2\epsilon,\theta,z), & p \in \Omega_R^{1-\epsilon} \setminus \Omega_R^{1-2\epsilon}; \\ u^\delta(1,\theta,z), & p \in \Sigma_R^\epsilon. \end{cases} \tag{18}$$

By Lemma 4.0.4, for each $\delta > 0$,

$$\big\|u^\delta\big\|_{L^2(\Omega_R;d\mu^\epsilon)} \to \big\|u^\delta\big\|_{L^2(\Omega_R;d\mu')} \tag{19}$$

as $\epsilon \to 0$. Taking into account that $\|u_\epsilon^\delta\|_{L^\infty} \leqslant \|u^\delta\|_{L^\infty}$ (this follows from the construction of u_ϵ^δ) and

$$d\mu^\epsilon = \begin{cases} d\mu, & \text{on } \Omega_R^{1-\epsilon}; \\ \frac{d\mu}{\epsilon}, & \text{on } \Sigma_R^\epsilon, \end{cases}$$

it follows that

$$\begin{aligned} \big\|u^\delta - u_\epsilon^\delta\big\|^2_{L^2(\Omega_R;,d\mu^\epsilon)} &= \int_{\Sigma_R^{2\epsilon}\setminus\Sigma_R^\epsilon} \big|u_\epsilon^\delta - u^\delta\big|^2\, d\mu^\epsilon + \int_{\Sigma_R^\epsilon} \big|u_\epsilon^\delta - u^\delta\big|^2\, d\mu^\epsilon \\ &\leqslant 2\big\|u^\delta\big\|^2_{L^\infty}\mu\big(\Sigma_R^{2\epsilon} \setminus \Sigma_R^\epsilon\big) + \int_{\Sigma_R^\epsilon} \big|u_\epsilon^\delta - u^\delta\big|^2 \frac{d\mu}{\epsilon}. \end{aligned}$$

The first term in the last line of this inequality clearly tends to 0 as $\epsilon \to 0$. We estimate the second term. By the mean value property, there exists $r' \in (r, 1)$ such that

$$\begin{aligned} \int_{\Sigma_R^\epsilon} \big|u_\epsilon^\delta - u^\delta\big|^2 \frac{d\mu}{\epsilon} &= \int_{\Sigma_R} d\sigma \int_{1-\epsilon}^{1} \big|u^\delta(1,\theta,z) - u^\delta(r,\theta,z)\big|^2 r\,\frac{dr}{\epsilon} \\ &\leqslant \int_{\Sigma_R} d\sigma \int_{1-\epsilon}^{1} \big|\big(u^\delta\big)_r\big(r',\theta,z\big)\big|^2 (1-r)^2 r\,\frac{dr}{\epsilon} \\ &\leqslant \big\|\big(u^\delta\big)_r\big\|^2_{L^\infty} \int_{\Sigma_R} d\sigma \int_{1-\epsilon}^{1} (1-r)^2 r\,\frac{dr}{\epsilon} \\ &= \big\|\big(u^\delta\big)_r\big\|^2_{L^\infty}\sigma(\Sigma_R)\frac{1}{\epsilon}\big[r^2/2 - (2/3)r^3 + r^4/4\big]_{1-\epsilon}^1 \to 0 \end{aligned}$$

as $\epsilon \to 0$. Thus, $u_\epsilon^\delta \to u^\delta$ in $L^2(\Omega_R; d\mu^\epsilon)$ as $\epsilon \to 0$. Applying the triangle inequality to this and (19), we find

$$\big\|u_\epsilon^\delta\big\|_{L^2(\Omega_R;d\mu^\epsilon)} \to \big\|u^\delta\big\|_{L^2(\Omega_R;d\mu')}$$

as $\epsilon \to 0$. Since we showed in the proof of (M1) in Theorem 3.0.4 that

$$\lim_{\epsilon\to 0} F_R^\epsilon\big[u_\epsilon^\delta\big] = F_R\big[u^\delta\big],$$

we may apply the diagonalization formula [1, Corollary 1.16] for (17) to find a mapping $\epsilon \to \delta(\epsilon)$ such that $\delta(\epsilon) \to 0$ as $\epsilon \to 0$ and

$$\begin{cases} \lim\limits_{\epsilon\to 0}\big\|u_\epsilon^{\delta(\epsilon)}\big\|_{L^2(\Omega_R;d\mu^\epsilon)} = \|u\|_{L^2(\Omega_R;d\mu')}; \\ \limsup\limits_{\epsilon\to 0} F_R^\epsilon\big[u_\epsilon^{\delta(\epsilon)}\big] \leqslant \lim\limits_{\delta\to 0}\Big(\lim\limits_{\epsilon\to 0} F_R^\epsilon\big[u_\epsilon^\delta\big]\Big) = F_R[u], \end{cases} \tag{20}$$

namely, $u_\epsilon := u_\epsilon^{\delta(\epsilon)}$ satisfies (iii).

Next, there exists $\tilde{u}^\delta \in C$ satisfying (i) by the definition of $\tilde{D}_R \subset C$.

Finally, in order to show (ii), we prove

$$\lim_{\epsilon\to 0}\big\|\tilde{u}^\delta - u_\epsilon\big\|_{L^2(\Omega_R;d\mu^\epsilon)} \leqslant \big\|\tilde{u}^\delta - u\big\|_{L^2(\Omega_R;d\mu')}, \tag{21}$$

indeed, this together with (i) will imply (ii). We estimate

$$\big\|\tilde{u}^\delta - u_\epsilon\big\|^2_{L^2(\Omega_R;d\mu^\epsilon)} = \int_{\Omega_R}\big(\tilde{u}^\delta\big)^2\,d\mu^\epsilon - 2\int_{\Omega_R}\tilde{u}^\delta u_\epsilon\,d\mu^\epsilon + \int_{\Omega_R}(u_\epsilon)^2\,d\mu^\epsilon. \tag{22}$$

Since $\tilde{u}^\delta \in C$, we may apply (16) and (20) to show that the first and third terms in (22) tend to

$$\int_{\Omega_R}\big(\tilde{u}^\delta\big)^2\,d\mu' \quad \text{and} \quad \int_{\Omega_R} u^2\,d\mu'$$

as $\epsilon \to 0$, respectively. Therefore, in order to show (21), it suffices to prove that the second term in (22) converges to $-2\int_{\Omega_R}\tilde{u}^\delta u\,d\mu'$. Since $d\mu^\epsilon = d\mu$ on $\Omega_R^{1-\epsilon}$,

$$\begin{aligned} &\bigg|\int_{\Omega_R}\tilde{u}^\delta u_\epsilon\,d\mu^\epsilon - \int_{\Omega_R}\tilde{u}^\delta u\,d\mu'\bigg| \\ &\quad\leqslant \int_{\Omega_R^{1-\epsilon}}\big|\tilde{u}^\delta\big||u_\epsilon - u|\,d\mu + \int_{\Sigma_R^\epsilon}\big|\tilde{u}^\delta u\big|\,d\mu \qquad (23)\\ &\qquad+\bigg|\int_{\Sigma_R^\epsilon}\tilde{u}^\delta u_\epsilon\,d\mu^\epsilon - \int_{\Sigma_R}\tilde{u}^\delta u\,d\sigma\bigg|. \qquad (24) \end{aligned}$$

First, we estimate the first term in (23). Since $u_\epsilon(r,\theta,z) = u_\epsilon^{\delta(\epsilon)}(r,\theta,z)$ is pinched by $u^{\delta(\epsilon)}(1,\theta,z)$ and $u^{\delta(\epsilon)}(r,\theta,z)$, it follows by setting $A_\epsilon = \Omega_R^{1-\epsilon}\setminus\Omega_R^{1-2\epsilon}$ that

$$\begin{aligned} \int_{\Omega_R^{1-\epsilon}}|u_\epsilon - u|\,d\mu &= \int_{A_\epsilon}\big|u_\epsilon^{\delta(\epsilon)} - u\big|\,d\mu + \int_{\Omega_R^{1-2\epsilon}}\big|u^{\delta(\epsilon)} - u\big|\,d\mu \\ &\leqslant \int_{A_\epsilon}\big|u^{\delta(\epsilon)}(1,\theta,z) - u\big|\,d\mu + \int_{A_\epsilon}\big|u^{\delta(\epsilon)} - u\big|\,d\mu + \int_{\Omega_R^{1-2\epsilon}}\big|u^{\delta(\epsilon)} - u\big|\,d\mu \\ &\leqslant \int_{A_\epsilon}\big|u^{\delta(\epsilon)}(1,\theta,z) - u\big|\,d\mu + \sqrt{\mu\big(\Omega_R^{1-\epsilon}\big)\int_{\Omega_R^{1-\epsilon}}\big|u^{\delta(\epsilon)} - u\big|^2\,d\mu} \\ &\leqslant \int_{1-2\epsilon}^{1-\epsilon} r\,dr\int_{\Sigma_R}\big|u^{\delta(\epsilon)}\big|_{\Sigma_R}\big|\,d\sigma + \int_{A_\epsilon}|u|\,d\mu + \sqrt{\mu\big(\Omega_R^{1-\epsilon}\big)\int_{\Omega_R^{1-\epsilon}}\big|u^{\delta(\epsilon)} - u\big|^2\,d\mu} \\ &\leqslant \epsilon C(R)\big\|u^{\delta(\epsilon)}\big\|_{L^2(\Omega_R;d\mu')} + \int_{A_\epsilon}|u|\,d\mu + \sqrt{\mu\big(\Omega_R^{1-\epsilon}\big)\int_{\Omega_R^{1-\epsilon}}\big|u^{\delta(\epsilon)} - u\big|^2\,d\mu} \to 0 \end{aligned}$$

as $\epsilon \to 0$ since $\mu(A_\epsilon) \to 0$. Thus, the first term in (23) tends to 0 as $\epsilon \to 0$. The second term in (23) tends to 0 because $\mu(\Sigma_R^\epsilon) \to 0$.

Finally, we estimate (24). Since

$$\int_{\Sigma_R} \tilde{u}^\delta u\, d\sigma = \lim_{\epsilon \to 0} \int_{\Sigma_R^\epsilon} \tilde{u}^\delta u(1, \theta, z)\, d\mu^\epsilon,$$

and $u_\epsilon(r, \theta, z) = u^{\delta(\epsilon)}(1, \theta, z)$ if $1 - \epsilon \leqslant r \leqslant 1$, it follows that

$$\begin{aligned}
\lim_{\epsilon \to 0} \left| \int_{\Sigma_R^\epsilon} \tilde{u}^\delta u_\epsilon\, d\mu^\epsilon - \int_{\Sigma_R} \tilde{u}^\delta u\, d\sigma \right| &\leqslant \lim_{\epsilon \to 0} \left| \int_{\Sigma_R^\epsilon} \tilde{u}^\delta u_\epsilon\, d\mu^\epsilon - \int_{\Sigma_R^\epsilon} \tilde{u}^\delta u(1, \theta, z)\, d\mu^\epsilon \right| \\
&\leqslant \lim_{\epsilon \to 0} \|\tilde{u}^\delta\|_{L^\infty} \int_{\Sigma_R^\epsilon} \left| u_\epsilon - u(1, \theta, z) \right| d\mu^\epsilon \\
&= \lim_{\epsilon \to 0} \|\tilde{u}^\delta\|_{L^\infty} \int_{\Sigma_R^\epsilon} \left| u^{\delta(\epsilon)}(1, \theta, z) - u(1, \theta, z) \right| d\mu^\epsilon \\
&= \lim_{\epsilon \to 0} \|\tilde{u}^\delta\|_{L^\infty} C(R) \left\| u^{\delta(\epsilon)}\big|_{\Sigma_R} - u|_{\Sigma_R} \right\|_{L^2(\Sigma_R; d\sigma)} \\
&= 0.
\end{aligned}$$

Now (21) is proved and we finished the proof for the case of $u \in D(F_R)$.

If $u \in L^2(\Omega_R; d\mu') \setminus D(F_R)$, then there exists $u_\epsilon \in L^2(\Omega_R; d\mu^\epsilon)$ which strong converges to u [12]. □

Remark 4.0.6. The same conclusion holds true for $R = \infty$, if we replace $\tilde{D}_R$ by $C_0^\infty(\overline{\Omega})$ in the proof.

4.0.2. Proof for (M2) condition

Proposition 4.0.7 *((M2) condition). Under the same condition in Theorem* 4.0.3*, for every $v_\epsilon \in L^2(\Omega_R; d\mu^\epsilon)$ weakly converging to $v \in L^2(\Omega_R; d\mu')$ in Kuwae–Shioya sense, it follows*

$$\liminf_{\epsilon \to 0} F_R^\epsilon[v_\epsilon] \geqslant F_R[v].$$

Proof. We show if $v_\epsilon \to v$ weakly in Kuwae–Shioya sense, then it does weakly converge in $L^2(\Omega_R; d\mu)$. Indeed, then by Theorem 3.0.4, it follows:

$$\liminf_{\epsilon \to 0} F_R^\epsilon[v_\epsilon] \geqslant F_R[v].$$

For an arbitrary $u \in L^2(\Omega_R; d\mu)$, set $\bar{u} \in L^2(\Omega_R; d\mu')$ and $u_\epsilon \in L^2(\Omega_R; d\mu^\epsilon)$ as

$$\bar{u}(p) = \begin{cases} u(p), & p \in \Omega_R \setminus \Sigma_R; \\ 0, & p \in \Sigma_R, \end{cases}$$

and

$$u_\epsilon = \frac{\bar{u}}{\omega^\epsilon}.$$

We can show that $u_\epsilon \in L^2(\Omega_R; d\mu^\epsilon)$ converges strongly to $\bar{u} \in L^2(\Omega_R; d\mu')$ as in the same argumentation in the proof of (M1) in the previous subsection. Therefore, if $v_\epsilon \in L^2(\Omega_R; d\mu^\epsilon)$ converges weakly to $v \in L^2(\Omega_R; d\mu')$ as $\epsilon \to 0$, then

$$\begin{aligned}
(v_\epsilon, u)_{d\mu} = (v_\epsilon, \bar{u})_{d\mu} &= \int_{\Omega_R} v_\epsilon \left(u_\epsilon \omega^\epsilon\right) d\mu \\
&= (v_\epsilon, u_\epsilon)_{d\mu^\epsilon} \to (v, \bar{u})_{d\mu'} = (v, u)_{d\mu}, \quad \text{as } \epsilon \to 0,
\end{aligned}$$

and we conclude that $v_\epsilon \to v$ weakly in $L^2(\Omega_R; d\mu)$ as $\epsilon \to 0$. □

5. Convergence of the finite-dimensional distributions

In Section 2, we showed that our energy functionals are local and regular Dirichlet forms, and by the Fukushima theorem [7], [8, Theorem 7.3.1], there are diffusion processes and Wiener measures associated to each of these Dirichlet forms. If the speed measure is fixed, then the semigroup convergence, which is a consequence of Mosco-convergence, implies the convergence of the finite-dimensional distributions of the Wiener measures (see e.g., the proof of Theorem 2.5 in [6]). Now we show that this is true also with changing speed measures. In this section, we drop the R for the sake of simplicity.

Let $\mathbb{P}^\epsilon$ and $\mathbb{P}'$ be the associated Wiener measures to $(F^\epsilon, D(F^\epsilon))$ in $L^2(\Omega; d\mu^\epsilon)$ and $(F, D(F))$ in $L^2(\Omega; d\mu')$ on the space $\mathcal{C} = C([0,\infty) \to \Omega)$ of continuous (in $t > 0$) trajectories on Ω such that

$$\mathbb{P}^\epsilon(\cdot) = \int_\Omega \mathbb{P}^\epsilon_p(\cdot)\, m(dp) \quad \text{and} \quad \mathbb{P}'(\cdot) = \int_\Omega \mathbb{P}'_p(\cdot)\, m(dp),$$

where dm is a probability measure on Ω. We impose Neumann boundary condition, and we assume that the corresponding semigroups T^ϵ_t and T'_t are conservative, i.e.,

$$T^\epsilon_t 1(p) = 1 = T'_t 1(p), \quad \text{for every } p \in \Omega \text{ and } t > 0.$$

(If T^ϵ_t and T'_t are not conservative, then we replace them by the conservative semigroups defined on the one-point compactification Ω_Δ. See e.g., [8]. However, we will prove that they are conservative later.)

For $0 \leqslant t_1 \leqslant \cdots \leqslant t_k$, define the projection $\pi_{t_1\cdots t_k} : \mathcal{C} \to \Omega^k$ by

$$\pi_{t_1\cdots t_k}(\omega) = \big(X(\omega, t_1), X(\omega, t_2), \ldots, X(\omega, t_k)\big).$$

This together with $\mathbb{P}^\epsilon$ ($\mathbb{P}'$, respectively) defines the probability measure on Ω^k, which is called the *finite-dimensional distribution* corresponding to $\mathbb{P}^\epsilon$ ($\mathbb{P}'$, respectively) (e.g., [4, p. 30]).

Lemma 5.0.8. *If $u_\epsilon \in L^2(\Omega; d\mu^\epsilon)$ converges strongly to $u \in L^2(\Omega; d\mu')$, then*

$$\int_\Omega v T^\epsilon_t u_\epsilon \, d\mu^\epsilon \to \int_\Omega v T'_t u \, d\mu', \quad \epsilon \to 0$$

for $v \in C$.

Proof. Because $v \in L^2(\Omega; d\mu^\epsilon)$ converges strongly to $v \in L^2(\Omega; d\mu')$ for $v \in C$. $\square$

Lemma 5.0.9. *Suppose $R < \infty$. If F^ϵ on $L^2(\Omega_R; d\mu^\epsilon)$ Mosco-converges to F on $L^2(\Omega_R; d\mu')$, then the finite-dimensional distribution corresponding to $\mathbb{P}^\epsilon$ weak converges to that of $\mathbb{P}'$ as $\epsilon \to 0$.*

Proof. Let $0 = t_0 < t_1 < t_2 < \cdots < t_n$ and $A_i \subset \Omega_R$ with $0 \leqslant i \leqslant n$. Set $\tau_i = t_i - t_{i-1}$. Denote by X^ϵ and X the processes associated to F^ϵ_R and F respectively, and by k^ϵ and k the associated transition functions (see e.g., [6, p. 156]). Define

$$\begin{aligned} f^\epsilon_n &= T^\epsilon_{\tau_n} \mathbf{1}_{A_n}, \\ f^\epsilon_i &= T^\epsilon_{\tau_i}\big(\mathbf{1}_{A_i} f^\epsilon_{i+1}\big), \quad 1 \leqslant i < n, \end{aligned}$$

and

$$\begin{aligned} f_n &= T'_{\tau_n} \mathbf{1}_{A_n}, \\ f_i &= T'_{\tau_i}(\mathbf{1}_{A_i} f_{i+1}), \quad 1 \leqslant i < n. \end{aligned}$$

By the fact that $F^\epsilon \to F'$ in Mosco sense and by applying Kuwae–Shioya theorem (Lemma 4.0.2),

$$f^\epsilon_i \to f_i \quad \text{strongly in Kuwae–Shioya sense for } 1 \leqslant i < n. \tag{25}$$

By the definition of k^ϵ,

$$\begin{aligned} \mathbb{E}\big[X^\epsilon_{t_i} \in A_i\colon 0 \leqslant i \leqslant n\big] &= \int_{A_0} \cdots \int_{A_{n-1}} k^\epsilon(\tau_n, p_{n-1}, A_n) k^\epsilon(\tau_{n-1}, p_{n-2}, dp_{n-1}) \cdots k^\epsilon(\tau_1, p_0, dp_1)\, d\mu^\epsilon \\ &= \int_{A_0} f^\epsilon_1 \, d\mu^\epsilon \\ &= \big(\mathbf{1}_{A_0}, f^\epsilon_1\big)_{L^2(\Omega_R; d\mu^\epsilon)} \end{aligned}$$

and the last expression strong converges in Kuwae–Shioya sense to

$$(\mathbf{1}_{A_0}, f_1)_{L^2(\Omega_R; d\mu')} = \mathbb{E}[X_{t_i} \in A_i\colon\ 0 \leqslant i \leqslant n]$$

as $\epsilon \to 0$ by (25). This completes the proof. □

6. Tightness

In this section, we show that the set of Wiener measures $\{\mathbb{P}^\epsilon\}_{\epsilon>0}$ associated to $(F_R^\epsilon, D(F_R^\epsilon))$ in $L^2(\Omega_R; d\mu^\epsilon)$ is tight; that is, for arbitrary $\delta > 0$ there exists a compact set $K \subset \mathcal{C}$ such that $\mathbb{P}^\epsilon(\mathcal{C} \setminus K) < \delta$ for every $\epsilon > 0$. The proof is to estimate the expectation of the displacement of the processes.

Theorem 6.0.10. *Let $0 < R \leqslant \infty$. If $\alpha \geqslant \max 2\{\beta, \gamma\}$, then the set of the Wiener measures $\{\mathbb{P}^\epsilon\}_{\epsilon>0}$ is tight.*

Proof. Let δ, h, and l be positive numbers and

$$\mathcal{C}_{h,l}^\delta = \Bigl\{\omega \in \mathcal{C}\colon \sup_{|t-s|<h;\ 0\leqslant t<s\leqslant l} \bigl|\omega(t) - \omega(s)\bigr| > \delta\Bigr\},$$

where $\mathcal{C}$ is the set of trajectories on Ω. Since F_R^ϵ is a local Dirichlet form, we may assume that the trajectories in $\mathcal{C}$ are continuous with time $t > 0$ almost surely (see e.g., [8]). We will show:

$$\lim_{h\to 0} \sup_{\epsilon>0} \mathbb{P}^\epsilon\bigl(\mathcal{C}_{h,l}^\delta\bigr) = 0, \tag{26}$$

which implies the tightness of $\{\mathbb{P}^\epsilon\}_{\epsilon>0}$ by [38, Theorem 6]. Because the process is nothing but a standard Brownian while it is in $\Omega^{1-\epsilon}$ and it moves faster when it is in Σ^ϵ, we need to estimate its displacement only on Σ^ϵ; in particular when $\epsilon \to 0$. In addition to this, we need show (26) only for the case $\nu = 0$, because it follows for every $h, l > 0$, and $\omega \in \mathcal{C}$ that

$$\sup_{|t-s|<h;\,0\leqslant t<s\leqslant l} \bigl|X_{\nu,\epsilon}(\omega, t) - X_{\nu,\epsilon}(\omega, s)\bigr| \leqslant \sup_{|t-s|<h;\,0\leqslant t<s\leqslant l} \bigl|X_\epsilon(\omega, t) - X_\epsilon(\omega, s)\bigr|,$$

where $X_{\nu,\epsilon}$ and X_ϵ, respectively, are the processes associated to F^ϵ defined in $L^2(\Omega; d\mu^\epsilon)$ and $L^2(\Omega; d\mu)$, respectively. Therefore, the operator which we need to study takes the form by the Stokes theorem:

$$A^\epsilon = \epsilon^{-\alpha} \frac{\partial_r(r\partial_r)}{r} + \epsilon^{-\beta}\left(\frac{\partial_\theta}{r}\right)^2 + \epsilon^{-\gamma}\partial_z^2 \tag{27}$$

on Σ^ϵ. The first term of the right-hand side of (27); namely, the derivative in the r-direction, is a 2-dimensional Bessel operator multiplied with the constant $\epsilon^{-\alpha}$ [9, p. 133], and the associated process $\mathbb{BES}^\alpha$, which is the r-direction of the process X_ϵ, is a 2-dimensional Bessel type process. We denote by $\mathbb{B}_\theta^\beta$ the θ-direction of the process X_ϵ and by $\mathbb{B}$ the linear Brownian motion. Since

$$\mathbb{B}_\theta^0(t) - \mathbb{B}_\theta^0(0) = \mathbb{B}\left(\int_0^t \frac{ds}{\mathbb{BES}^0(s)}\right) \pmod{2\pi}$$

and taking into account that ϵ is close to 0; namely, $\mathbb{BES}^0(s)$ is close to 1, we find a constant $C > 0$ such that

$$\bigl|\mathbb{B}_\theta^\beta(s) - \mathbb{B}_\theta^\beta(t)\bigr| \leqslant C\epsilon^{-\beta/2}\bigl|\mathbb{B}(s) - \mathbb{B}(t)\bigr| \quad \text{for } s, t > 0 \pmod{2\pi},$$

therefore, we need to estimate

$$\mathbb{E}\left[\sup_{0<\sigma<\tau}\left(\int_0^\tau \epsilon^{-\beta/2}\mathbf{1}_{[1-\epsilon,1]}\bigl(\mathbb{BES}^\alpha(s)\bigr)\,\mathbb{B}(ds)\right)\right].$$

The scale function σ and the speed measure m for $\mathbb{BES}^\alpha$ with reflecting boundary condition at $r = 1$ (see [9, Chapter 5] and [31]) are

$$\sigma(r) = \ln(r), \quad 0 < r \leqslant 1,$$

and

$$m(r) = \begin{cases} (\epsilon^\alpha/2)\exp(2r), & r \leqslant 0, \\ \epsilon^\alpha/2, & r > 0, \end{cases}$$

so

$$m(dr) = \begin{cases} \epsilon^{\alpha} \exp(2r)\, dr, & r \leqslant 0, \\ 0, & r > 0. \end{cases}$$

If we denote by $l(t,\xi)$ the Brownian local time at time t and position ξ, and if we define

$$h(t) := \int_{\mathbb{R}} l(t,\xi)\, m(d\xi) = \epsilon^{\alpha} \int_{-\infty}^{0} l(t,\xi) e^{2\xi}\, d\xi,$$

then the stochastic clock (the time substitution) [9, p. 165], [37] of $\mathbb{BES}^{\alpha}$ is $t \to h^{-1}(t)$, and it follows (see e.g., [32, Theorem 47.1]) that

$$\left(\mathbb{BES}^{\alpha}, \mathbb{P}_r\right) \equiv \left(\exp\left(\mathbb{B}\left(h^{-1}\right)\right), \mathbb{P}_{\ln(r)}\right) \quad \text{(in law)}. \tag{28}$$

Since

$$h(t) = \int_{-\infty}^{0} l(t,\xi) e^{2\xi}\, dx = \epsilon^{\alpha} \int_{0}^{t} \exp\left(2\mathbb{B}(s)\right) ds$$

(see e.g., [20, p. 33]), and

$$h^{-1}(\tau) \leqslant \tau \epsilon^{-\alpha}/2,$$
$$\dot{h}(t) = \epsilon^{\alpha} \exp\left(2\mathbb{B}(t)\right),$$

it follows by (28), Doob's inequality (as in [36, p. 63]), and the change of variable: $t = h^{-1}(s)$ that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0<\sigma<\tau}\left(\int_0^{\sigma} \epsilon^{-\beta/2} \mathbf{1}_{[1-\epsilon,1]}\left(\mathbb{BES}^{\alpha}(s)\right) \mathbb{B}(ds)\right)^2\right] \\
&\quad = \epsilon^{-\beta} \mathbb{E}\left[\sup_{0<\sigma<\tau}\left(\int_0^{\sigma} \mathbf{1}_{[\ln(1-\epsilon),0]}\left(\mathbb{B}\left(h^{-1}(s)\right)\right) \mathbb{B}(ds)\right)^2\right] \\
&\quad \leqslant 4\epsilon^{-\beta} \mathbb{E}\left[\int_0^{\tau} \mathbf{1}_{[\ln(1-\epsilon),0]}\left(\mathbb{B}\left(h^{-1}(s)\right)\right) ds\right] \\
&\quad \leqslant 4e^2 \epsilon^{\alpha-\beta} \mathbb{E}\left[\int_0^{h^{-1}(\tau)} \mathbf{1}_{[\ln(1-\epsilon),0]}\left(\mathbb{B}(t)\right) dt\right] \\
&\quad \leqslant 4e^2 \epsilon^{\alpha-\beta} \int_{\mathbb{R}} \int_0^{\tau\epsilon^{-\alpha}/2} \mathbf{1}_{[\ln(1-\epsilon),0]}(x) \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\, dt\, dx \\
&\quad = 4e^2 \epsilon^{\alpha-\beta} \int_{\ln(1-\epsilon)}^{0} \int_1^{\tau\epsilon^{-\alpha}/2} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\, dt\, dx \\
&\qquad + 4e^2 \epsilon^{\alpha-\beta} \int_{\ln(1-\epsilon)}^{0} \int_0^{1} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\, dt\, dx = (\mathrm{I}) + (\mathrm{II}).
\end{aligned}$$

The second term, (II), tends to 0 as $\epsilon \to 0$ provided $\alpha \geqslant \beta$ by Lebesgue's dominated convergence theorem. The first term, (I), can be estimated as

$$\begin{aligned}
(\mathrm{I}) &\leqslant C\epsilon^{\alpha-\beta} \int_{\ln(1-\epsilon)}^{0} \int_1^{\tau\epsilon^{-\alpha}/2} \frac{1}{\sqrt{t}}\, dt\, dx \\
&\leqslant -2C\epsilon^{\alpha-\beta} \ln(1-\epsilon)\sqrt{\tau\epsilon^{-\alpha}/2} \\
&= -2C\epsilon^{\alpha/2-\beta} \ln(1-\epsilon)\sqrt{\tau/2},
\end{aligned}$$

with a positive constant C which is independent of ϵ and τ, and the last line tends to 0 as $\epsilon \to 0$ provided $\alpha \geqslant 2\beta$. We deduce that

$$\lim_{\tau\to 0}\sup_{\epsilon>0}\mathbb{E}\left[\sup_{0<\sigma<\tau}\left(\int_0^\sigma \epsilon^{-\beta/2}\mathbf{1}_{[1-\epsilon,1]}\big(\mathbb{BES}^\alpha(s)\big)\,\mathbb{B}(ds)\right)\right]=0.$$

In a similar way, we can estimate the processes in the z-direction provided $\alpha \geqslant 2\gamma$.

Finally, denoting by X_ϵ^r the r-component of X_ϵ,

$$\mathbb{P}^\epsilon\big(\big|X_\epsilon^r(t)\big|>\delta\big)\leqslant \mathbb{P}\big(\big|\mathbb{B}^r(t)\big|>\delta/2\big),\quad \text{if } \epsilon<\delta/2,$$

since X_ϵ behaves on $\Omega^{1-\epsilon}$ as the standard Brownian motion. We deduce

$$\lim_{h\to 0}\sup_{\epsilon>0}\mathbb{P}^\epsilon\big(\mathcal{C}_{h,l}^\delta\big)=0$$

implying that $\{\mathbb{P}^\epsilon\}_{\epsilon>0}$ is tight. □

7. Extension to unbounded domain

In this section, we extend the previous results of Mosco-convergence to the unbounded domain Ω. As another issue, we also discuss the conservation property and the recurrence of the processes.

Lemma 7.0.11. *If $\alpha \geqslant 0$, $0 \leqslant \beta, \gamma, \nu \leqslant 1$, and $\alpha \geqslant \max 2\{\beta,\gamma\}$, then the finite-dimensional distribution of $\mathbb{P}^\epsilon$ converges to that of $\mathbb{P}$.*

Proof. Let $0<t_1<t_2<\cdots<t_n$ and $A_i \subset \Omega$ be open sets with $1<i<n$. Due to the estimate in the proof of Theorem 6.0.10, for any $\delta>0$ there exists R_δ such that for $R>R_\delta$

$$\sup_{\epsilon>0}\mathbb{P}^\epsilon(t_i\geqslant\tau_R)+\mathbb{P}(t_i\geqslant\tau_R)<\delta, \tag{29}$$

for every $1\leqslant i\leqslant n$, where τ_R is the exit time for Ω_R. Setting $\Lambda=\{X_{t_1}\in A_1,\ X_{t_2}\in A_2,\ \ldots,\ X_{t_n}\in A_n\}$, taking into account that the processes associated to F_R^ϵ and F^ϵ are the same before they leave Ω_R, it follows by (29) that

$$\begin{aligned}\big|\mathbb{P}^\epsilon(\Lambda)-\mathbb{P}(\Lambda)\big| &\leqslant \big|\mathbb{P}^\epsilon(\Lambda)-\mathbb{P}^\epsilon\big(\Lambda\cap\{t_i<\tau_R\}\big)\big|+\big|\mathbb{P}^\epsilon\big(\Lambda\cap\{t_i<\tau_R\}\big)-\mathbb{P}\big(\Lambda\cap\{t_i<\tau_R\}\big)\big|\\ &\quad+\big|\mathbb{P}(\Lambda)-\mathbb{P}\big(\Lambda\cap\{t_i<\tau_R\}\big)\big|\\ &\leqslant \mathbb{P}^\epsilon(t_i\geqslant\tau_R)+\big|\mathbb{P}^\epsilon\big(\Lambda\cap\{t_i<\tau_R\}\big)-\mathbb{P}\big(\Lambda\cap\{t_i<\tau_R\}\big)\big|+\mathbb{P}(t_i\geqslant\tau_R)\\ &\leqslant \big|\mathbb{P}^\epsilon\big(\Lambda\cap\{t_i<\tau_R\}\big)-\mathbb{P}\big(\Lambda\cap\{t_i<\tau_R\}\big)\big|+\delta.\end{aligned}$$

The last line tends to δ as $\epsilon\to 0$, because $\mathbb{P}^\epsilon(\Lambda\cap\{t_i<\tau_R\})$ is the finite-dimensional distribution associated to F_R^ϵ which converges to $\mathbb{P}(\Lambda\cap\{t_i<\tau_R\})$. Since $\delta>0$ is arbitrary, we arrive at the conclusion. □

Remark 7.0.12. Our approach in Lemma 7.0.11 is inspired by [35] and [15].

Combining Lemma 5.0.9, Theorem 6.0.10, and Lemma 7.0.11, we obtain

Theorem 7.0.13. *If $\alpha \geqslant 0$, $0 \leqslant \beta, \gamma, \nu \leqslant 1$, and $\alpha \geqslant \max 2\{\beta,\gamma\}$, then the Wiener measure $\mathbb{P}^\epsilon$ converges weakly to $\mathbb{P}$ as $\epsilon \to 0$.*

The fact that $\mathbb{P}^\epsilon \to \mathbb{P}$ weakly implies the Mosco-convergence can be seen as follows.

Theorem 7.0.14. *Under the condition in Theorem* 7.0.13, *it follows*

$$F^\epsilon \to F \quad \textit{in Mosco sense.}$$

Proof. First, let $\nu\equiv 0$. Let $A\subset\Omega$ be any open set and denote by $\mathbf{1}_A$ its indication function. Lemma 7.0.11 implies

$$T_t^\epsilon\mathbf{1}_A(p)=\mathbb{P}_p^\epsilon\big(X^\epsilon(t)\in A\big)\to\mathbb{P}_p\big(X(t)\in A\big)=T_t\mathbf{1}_A(p),\quad \text{as } \epsilon\to 0$$

for quasi every $p\in\Omega$. Since T_t and T_t^ϵ are Markov semigroups,

$$\big\|T_t^\epsilon\mathbf{1}_A-T_t\mathbf{1}_A\big\|_{L^\infty(\Omega)}\leqslant 1.$$

Thus, we can apply Lebesgue's dominated convergence theorem to deduce

$$T_t^\epsilon\mathbf{1}_A\to T_t\mathbf{1}_A \quad \text{in } L^2(\Omega;d\mu),\ \text{as } \epsilon\to 0.$$

For $u \in L^2(\Omega; d\mu)$ and $\delta > 0$, let v be the step function approximating u as $\|u - v\|_{L^2(\Omega;d\mu)} < \delta/2$. By the contractive property of T_t^ϵ and T_t,

$$\left\|T_t^\epsilon(u-v)\right\|_{L^2(\Omega;d\mu)} + \left\|T_t(u-v)\right\|_{L^2(\Omega;d\mu)} < \delta.$$

Thus,

$$\begin{aligned}&\left\|T_t^\epsilon u - T_t u\right\|_{L^2(\Omega;d\mu)}\\ &\quad \leqslant \left\|T_t^\epsilon(u-v)\right\|_{L^2(\Omega;d\mu)} + \left\|T_t^\epsilon v - T_t v\right\|_{L^2(\Omega;d\mu)} + \left\|T_t(v-u)\right\|_{L^2(\Omega;d\mu)},\end{aligned}$$

where the last line is bounded by δ at the limit $\epsilon \to 0$. Since $\delta > 0$ is arbitrary, and by Mosco theorem, this is equivalent to the Mosco-convergence.

Next, let $0 < \nu \leqslant 1$. (M1) condition holds true in this case (see Remarks 3.1.2 and 4.0.6) and we prove only (M2). By applying the argumentation in the proof of Proposition 4.0.7 ((M2) with $0 < \nu \leqslant 1$ for $R < \infty$), we deduce (M2) for the current setting. □

We show the conservation property of the processes on Ω.

Theorem 7.0.15. *Under the condition in Theorem* 7.0.13*, the processes associated to F^ϵ defined in $L^2(\Omega; d\mu^\epsilon)$, F defined in $L^2(\Omega; d\mu)$ and $L^2(\Omega; d\mu')$ are conservative. More strongly, they are recurrent.*

Proof. First, we show the conservation property of F in $L^2(\Omega; \mu)$. By the estimate of $|X^\epsilon|$ in the proof of Theorem 6.0.10, for any $\delta > 0$ there exists $l_\delta > 0$ such that for $l > l_\delta$

$$\lim_{\epsilon\to 0}\left(1 - T_1^\epsilon \mathbf{1}_{\Omega_l}(p)\right) = \lim_{\epsilon\to 0}\mathbb{P}_p^\epsilon\left(X^\epsilon(1) \notin \Omega_l\right) \leqslant \lim_{\epsilon\to 0}\mathbb{P}_p^\epsilon\left(\left|X^{\epsilon,z}(1)\right| > l\right) < \delta, \tag{30}$$

for all $p \in \Omega$. Let $L > 0$ be arbitrary. Since the Mosco-convergence implies the semigroup convergence in L^2, it follows

$$\lim_{\epsilon\to 0}\left\|\left(T_1^\epsilon - T_1\right)\mathbf{1}_{\Omega_l}\right\|_{L^2(\Omega_L;d\mu)} \leqslant \lim_{\epsilon\to 0}\left\|\left(T_1^\epsilon - T_1\right)\mathbf{1}_{\Omega_l}\right\|_{L^2(\Omega;d\mu)} = 0.$$

Combining this together with (30), we deduce

$$\begin{aligned}\|1 - T_1 1\|_{L^2(\Omega_L;d\mu)} &\leqslant \|1 - T_1\mathbf{1}_{\Omega_l}\|_{L^2(\Omega_L;d\mu)}\\ &\leqslant \lim_{\epsilon\to 0}\left\|1 - T_1^\epsilon\mathbf{1}_{\Omega_l}\right\|_{L^2(\Omega_L;d\mu)} + \lim_{\epsilon\to 0}\left\|T_1^\epsilon\mathbf{1}_{\Omega_l} - T_1\mathbf{1}_{\Omega_l}\right\|_{L^2(\Omega_L;d\mu)}\\ &\leqslant \delta\mu(\Omega_L).\end{aligned}$$

Because $\delta > 0$ and $L > 0$ are arbitrary, this shows that

$$T_1 1 \equiv 1, \quad \mu\text{-a.e. on } \Omega.$$

By the semigroup property, $T_t T_{1-t} 1 = T_1 1 = 1$ for every $0 < t < 1$, and we deduce

$$T_t 1 \equiv 1, \quad \mu\text{-a.e. on } \Omega$$

for every $0 < t < 1$, and hence for every $t > 0$; that is, the conservation property of the process.

Next we consider the same problem for F defined in $L^2(\Omega; d\mu')$. By the Mosco-convergence of $F^\epsilon \to F$ in Kuwae–Shioya sense, there exists $\tilde{u}_\delta \in C_0(\overline{\Omega})$ such that

$$\lim_{\delta\to 0}\|\tilde{u}_\delta - T_1\mathbf{1}_{\Omega_l}\|_{L^2(\Omega;d\mu')} = 0 \tag{31}$$

and

$$\lim_{\delta\to 0}\limsup_{\epsilon\to 0}\left\|\tilde{u}_\delta - T_1^\epsilon\mathbf{1}_{\Omega_l}\right\|_{L^2(\Omega;d\mu^\epsilon)} = 0 \tag{32}$$

for any fixed $l > 0$. On the other hand, for any $\hat{\delta} > 0$, there exists $l_{\hat{\delta}} > 0$ such that for every $l > l_{\hat{\delta}}$

$$\lim_{\epsilon\to 0}\left(1 - T_1^\epsilon\mathbf{1}_{\Omega_l}(p)\right) < \hat{\delta} \tag{33}$$

for every $p \in \Omega$. Let $L > 0$ be arbitrary.

$$\begin{aligned}&\|1 - T_1 1\|_{L^2(\Omega_L;d\mu')}\\ &\quad\leqslant \limsup_{\epsilon\to 0}\left\|1 - T_1^\epsilon\mathbf{1}_{\Omega_l}\right\|_{L^2(\Omega_L;d\mu')} + \limsup_{\epsilon\to 0}\left\|T_1^\epsilon\mathbf{1}_{\Omega_l} - T_1\mathbf{1}_{\Omega_l}\right\|_{L^2(\Omega_L;d\mu')}.\end{aligned}$$

By applying (33), the first term of the second line of this inequality can be estimated from above by $\hat{\delta}(\mu'(\Omega_L))^{1/2}$. The second term is estimated from above by applying (31) and (33) as

$$\begin{aligned}
&\lim_{\delta\to 0}\limsup_{\epsilon\to 0}\big\|T_1^\epsilon \mathbf{1}_{\Omega_L}-\tilde{u}_\delta\big\|_{L^2(\Omega_L;d\mu')}+\lim_{\delta\to 0}\|\tilde{u}_\delta - T_1\mathbf{1}_{\Omega_L}\|_{L^2(\Omega_L;d\mu')}\\
&\quad\leqslant \limsup_{\epsilon\to 0}\big\|T_1^\epsilon \mathbf{1}_{\Omega_L}-1\big\|_{L^2(\Omega_L;d\mu')}+\lim_{\delta\to 0}\|1-\tilde{u}_\delta\|_{L^2(\Omega_L;d\mu')}\\
&\quad\leqslant \hat{\delta}\big(\mu'(\Omega_L)\big)^{1/2}+\lim_{\delta\to 0}\|1-\tilde{u}_\delta\|_{L^2(\Omega_L;d\mu')}.
\end{aligned}$$

Since $\tilde{u}_\delta$ is continuous and by (32), the second term of the last line in the above inequality is estimated as:

$$\begin{aligned}
&\lim_{\delta\to 0}\limsup_{\epsilon\to 0}\|1-\tilde{u}_\delta\|_{L^2(\Omega_L;d\mu^\epsilon)}\\
&\quad\leqslant \limsup_{\epsilon\to 0}\big\|1-T_1^\epsilon \mathbf{1}_{\Omega_L}\big\|_{L^2(\Omega_L;d\mu^\epsilon)}+\lim_{\delta\to 0}\limsup_{\epsilon\to 0}\big\|T_1^\epsilon \mathbf{1}_{\Omega_L}-\tilde{u}_\delta\big\|_{L^2(\Omega_L;d\mu^\epsilon)}\\
&\quad= \limsup_{\epsilon\to 0}\big\|1-T_1^\epsilon \mathbf{1}_{\Omega_L}\big\|_{L^2(\Omega_L;d\mu^\epsilon)}\\
&\quad\leqslant \hat{\delta}\Big(\limsup_{\epsilon\to 0}\mu^\epsilon(\Omega_L)\Big)^{1/2}=\hat{\delta}\big(\mu'(\Omega_L)\big)^{1/2}.
\end{aligned}$$

Summing up those estimates, we obtain

$$\|1-T_1 1\|_{L^2(\Omega_L;d\mu')}\leqslant 3\hat{\delta}\big(\mu'(\Omega_L)\big)^{1/2},$$

and again by the semigroup property, we conclude that T_t is conservative in $L^2(\Omega;d\mu')$.

Next, we show the recurrence of F in $L^2(\Omega;d\mu)$ and $L^2(\Omega;d\mu')$. For $\rho>0$, let

$$\chi_\rho(r,\theta,z)=\big(1\wedge\big(2-|z|/\rho\big)\big)_+.$$

Then $\chi_\rho\in D(F)$ and

$$F[\chi_\rho]=\pi\int_\rho^{2\rho}\rho^{-2}\,dz\to 0,\quad \text{as } \rho\to\infty.$$

Since $\chi_\rho\to 1$ a.e. in both measures μ and μ', we may apply Oshima's recurrent criteria [29,8] to conclude that the process is recurrent.

For fixed $\epsilon>0$, it follows

$$F^\epsilon[\chi_\rho]\to 0,\quad \text{as } \rho\to\infty,$$

and we deduce the recurrence for the process associated to F^ϵ in $L^2(\Omega;d\mu^\epsilon)$. Since the recurrence implies the conservation property of the process [29,8], we complete the proof. □

Remark 7.0.16. As we stated in the introduction, the conservation property follows from the recurrence; however we independently proved them. This is because if we consider a more general setting such as $B\times\mathbb{R}^n$ or a Riemannian manifold with boundary, then there are many important manifolds whose processes are not recurrent but merely conservative; for instance, if we replace $\mathbb{R}$ in our setting by $\mathbb{R}^n$, then our proof shows that the associated processes are recurrent if and only if $n=1,2$; and are conservative for every $n\geqslant 1$.

Remark 7.0.17. Our approach to conservation property in Theorem 7.0.15 is inspired by [2].

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